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THE JET-WAVE

THEORY OF THE PERIODIC JET-WAVE

BY

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PREFACE

The present treatise contains a series of theoretical investigations on the properties of the jet-wave. The latter has during recent years achieved a certain significance through its various applications.[1] A special importance may be ascribed to the periodic jet-wave owing to the predominant part it plays in the jet-wave rectifier, the first high capacity mechanical rectifier ever produced.[2] The present paper in the main confines itself to this type of wave. In the first chapter the theory of a wave of small amplitude is considered. The latter theory permits a rather complete discussion of the properties of the wave under various conditions. Some of its results have already been stated in a previous Danish treatise (Nye Ensrettere og periodiske Afbrydere). For the sake of completeness and in order to have the said results

[1] The Jet-wave and its Applications. Paper read before Section G of the British Association at Glasgow, September 11, 1928. "Engineering" Sept. 14, 1928.

[2] 1) Nye Ensrettere og periodiske Afbrydere. Jul. Gjellerups Forlag, København 1918.

2) Development of the Jet-Wave Rectifier. Paper read before Section G of the British Association at Leeds, September 5, 1927. "Engineering" September 9 and 16, 1927.

3) Den konstruktive Udvikling af Straalebølgeensretteren. "Elektroteknikeren" Nr. 23 1927.

4) Güntherschulze: Die konstruktive Durchbildung des Quecksilber-Wellenstrahl-Gleichrichters. Elektrotechnische Zeitschrift 16. August 1928.

presented in a language generally known, the theory has here been given at full length, including the earlier results and some additional discussions. — In chapter II the complete theory of a jet-wave of partly arbitrary amplitude is represented and in addition it is shown how an approximate theory sufficient in most practical cases may be produced. Finally graphical methods for the production of pictures of the wave are considered. — In chapter III special properties of the jet-wave are made the subject of investigation and formulae for the electric resistance of the wave, for the heating of the same by an electric current etc. are derived. Finally, in an appendix, a preliminary test on the statements of the theory has been given.

In conclusion I desire to express my thanks to the Trustees of the Carlsberg Fund for having enabled me to take the time necessary for the completion of the work.

Physical Laboratory II, The Royal Technical College.
Copenhagen, October 1928.

Jul. Hartmann.

CHAPTER I

The Jet-Wave of small Amplitude.

1. The Jet-Wave.

If the nozzle N of a liquid jet is moved to and fro perpendicularly to the axis of the nozzle, the jet assumes the shape of the wave-line indicated in fig. 1. The jet thus deformed is called a jet-wave. In the case considered the wave is produced in the following way. The individual jet-particle will, in passing the nozzle, assume the velocity of the same and keep it on its way onward together with its original velocity i. e. that of the original jet. It will therefore follow a straight line, say ab in fig. 1, which forms a certain angle θ with the direction of the axis of the original jet depending on the velocity of the nozzle at the moment of departure of the particle. The direction thus varies from particle to particle. If the nozzle performs harmonic oscillations, the direction will also vary in a har

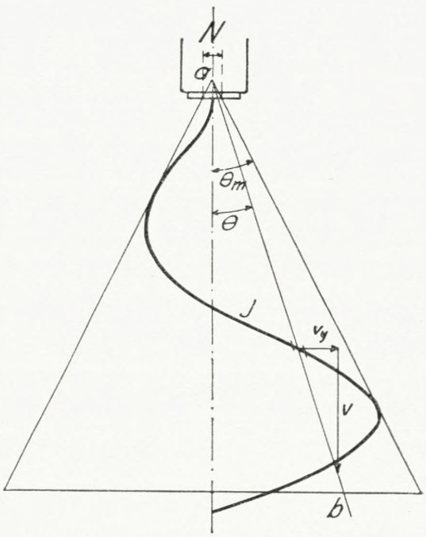


Fig. 1. Jet-Wave produced by an oscillating Nozzle.

monic way and the consecutive particles must arrange themselves along a simple wave-line, the amplitude of which increases proportionally to the distance from the nozzle (or nearly so). The jet-wave moves on as a unity with the velocity v of the original jet. Its wave-length is obviously determined by

$$\lambda = v \cdot T, \tag{1}$$

if T is the period of the oscillations of the nozzle.

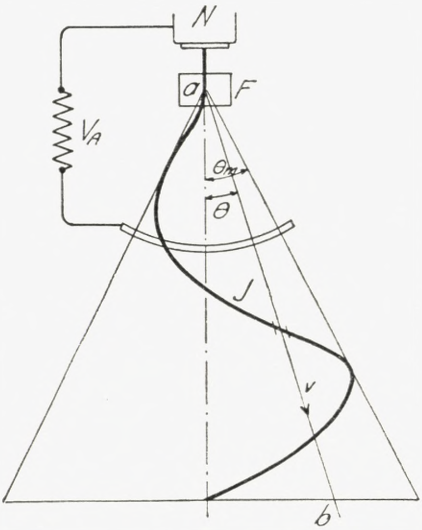


Fig. 2. Jet-Wave electromagnetically produced.

Now, if the jet is made of an electrically conductive liquid, say mercury, a jet-wave may be produced in another very simple way indicated in fig. 2. The jet J passes a constant magnetic field F, the lines of force of which are perpendicular to the jet and in the figure also to the plane of the picture. An alternating current, the auxiliary current, supplied by a suitable transformer V_A, is passed through that part of the jet which is inside the field at any time. The current may be led into and out of the jet through the nozzle and a special electrode, the auxiliary electrode, touching the jet. Owing to the interaction between the current and the field the consecutive particles will be attacked by a periodic, mechanical force which is nearly perpendicular to the jet and to the magnetic field, thus situated in the plane of the picture. Accordingly they will be sent out along a line of a direction θ varying periodically with the time as in the former case, and consequently a similar wave will be produced. Obviously, instead of employing a constant field and an alternating current, an alternating field in interaction with a constant current may as well be employed for the production of the wave. The subject of the present paper is mainly the theory of the electromagnetically produced periodic jet-wave.

2. The Jet-Wave with small Amplitude and a laminar Field.

In the first instance we shall confine ourselves to waves of such small amplitude that the mechanical force produced through the interaction of field and current may be considered as perpendicular to the axis of the original jet during the whole passage of a particle through the field. Furthermore we will assume that the extension *dl* fig. 3 of the field in the direction of the original jet is small compared to the wave-length. This is the same as to assume that the current may be considered constant during the passage of a small particle Δx of the jet. Finally we shall base the following theory on the supposition of the individual particles moving independently of each other. We shall thus neglect the cohesion and friction between the particles of the jet.

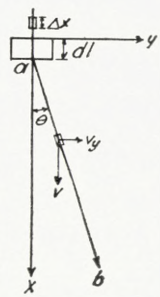


Fig. 3. Theory of the Jet-Wave with small Amplitude.

The mechanical force acting on the particle Δx passing the field at the moment t_0 is

$$(1) \qquad K = \frac{1}{10} i_{t_0} \cdot H_{t_0} \cdot \Delta x$$

provided H_{t_0} is the intensity of the field and i_{t_0} the value of the current at the said moment. We shall thus preliminarily assume that both vary with the time. As indicated the force is perpendicular to the original jet, that is to say, to the x-axis in fig. 3. During the passage it will give the particle considered a velocity v_y perpendicular to the said axis determined by

$$m \cdot \Delta x \cdot v_y = K \cdot \frac{dl}{v} \tag{2}$$

m indicating the mass per cm of the original jet, and $\frac{dl}{v}$ being the time which the passage takes. From (1) and (2) is found

$$v_y = \frac{1}{10} \cdot i_{t_0} \cdot H_{t_0} \cdot \frac{dl}{m \cdot v}. \tag{3}$$

The direction of the path of the particle after the field is left is determined by

$$tg\,\theta = \frac{v_y}{v} = \frac{1}{10} \cdot i_{t_0} \cdot H_{t_0} \cdot \frac{dl}{m\,v^2}. \tag{4}$$

The coordinates of the particle at the moment t $(t > t_0)$ are

$$x = v\,(t - t_0) \tag{5}$$

and

$$y = v_y \cdot (t - t_0) \tag{6}$$

from which

$$y = \frac{v_y}{v} \cdot x = \frac{1}{10}\, i_{t_0} \cdot H_{t_0} \cdot \frac{dl}{m\,v^2} \cdot x. \tag{7}$$

If the field is constant and its intensity equal to H and if the current varies with time according to

$$i = f(t) \tag{8}$$

(7) becomes

$$y = \frac{1}{10}\, \frac{H\,dl}{m\,v^2} \cdot x \cdot f(t_0). \tag{9}$$

The equation of the wave line at an arbitrary moment t is found by eliminating t_0 from (9) and (5). It is

(10) $$y = \frac{1}{10}\frac{H\,dl}{m\,v^2}\cdot x\cdot f\left(t-\frac{x}{v}\right).$$

In the following way it is seen from (10) that the wave proceeds in the direction of the x-axis with the velocity v, thus the velocity of the original jet. The intersection with the x-axis is determined by

(11) $$f\left(t-\frac{x}{v}\right) = 0.$$

If now $t-\frac{x}{v} = a$ is a value satisfying (11), it follows that after the lapse of dt the point of intersection has been moved forward by dx where

(12) $$dt-\frac{dx}{v} = 0.$$

from which

(13) $$\frac{dx}{dt} = v.$$

The expression (10) may also be looked on as representing the motion of the point of intersection between the jet-wave and a line or plane perpendicular to the x-axis at a distance x from the field. It is seen that the said motion is given by just the same function of time as the current, only it is delayed by $\frac{x}{v}$ seconds in relation to the current.

The motion thus pictures the current. On this fact the jet-wave oscillograph is based. In the latter an image of the jet-wave is projected on to a wall parallel to the plane of the wave. In the wall is a slit perpendicular to the axis of the original jet. Behind the slit a photographic plate or film is moved with constant velocity in a direction perpendicular to the slit. On the plate the projection thus traces a picture of the current passed through the jet.

3. The Wave produced by a simple alternating Current.

We shall now consider some few particular cases. In the first instance we shall assume the wave to be produced by a simple alternating current.

$$i = I \sin \omega t = I \sin \frac{2\pi}{T} t. \tag{1}$$

The equation of the jet-wave then becomes

$$y = \frac{1}{10} \frac{HI}{m v^2} \cdot dl \cdot x \cdot \sin \omega \left(t - \frac{x}{v} \right). \tag{2}$$

The expression obviously represents a sine-shaped wave fig. 4 which travels on with the velocity v, the amplitude at the same time increasing proportionally to the distance from the field, that is to say, the starting-point of the wave. During its motion the wave-tops touch the two lines

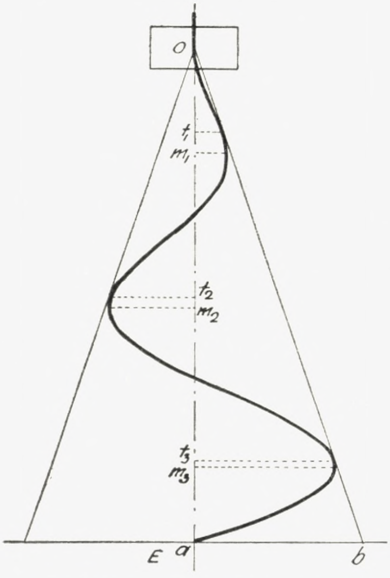


Fig. 4. Wave produced by a simple alternating Current.

$$y = \pm \frac{1}{10} \frac{HI}{m v^2} \cdot dl \cdot x, \tag{3}$$

which form an angle θ_m with the axis of the original jet given by

$$tg\, \theta_m = \pm \frac{1}{10} \frac{HI}{m v^2} \cdot dl = \pm \alpha. \tag{4}$$

Obviously (3) represents the lines which the jet would stationarily follow if direct currents $\pm I$, thus currents equal to the maximum value of the alternating current, were sent through the jet. Generally $tg\, \theta_m$ found from (4) is taken to measure the amplitude of the jet-wave and is denoted by α. If introduced in (2) this latter expression becomes

(5) $$y = \alpha x \cdot \sin \omega \left(t - \frac{x}{v}\right),$$

while the equations of the lines limiting the wave track may be written

(6) $$y = \pm \alpha x.$$

From (5) is seen that the wave will, at a given moment, cut the x-axis in a series of points, the zero-points, determined by

(7) $$\sin \omega \left(t - \frac{x}{v}\right) = 0.$$

Obviously these points are situated at a distance from each other given by

(8) $$\frac{\lambda}{2} = v \cdot \frac{T}{2}.$$

λ is called the wave-length of the wave. On the other hand it is seen from (5) and (6) that the points t_1, t_2, t_3, fig. 4, at which the wave touches the lines (6) are determined by

(9) $$\sin \omega \left(t - \frac{x}{v}\right) = 1.$$

They are thus to be found half-way between the consecutive zero-points of the wave. The tops of the wave, m_1, m_2, m_3, fig. 4, that is to say the points at which $\frac{dy}{dx} = 0$, are situated a little farther on in the direction of the motion. In order to see this we may preferably consider the wave at the moment $t = 0$. At this juncture $\frac{dy}{dx} = 0$ at the points determined by

(10) $$tg \frac{\omega x}{v} = -\frac{\omega x}{v}.$$

As known, this equation is solved graphically by finding the point of intersection between the curves

$y = tg\frac{\omega x}{v}$ and $y = -\frac{\omega x}{v}$. It is seen that the solutions are given by

$$\frac{\omega x}{v} = p\cdot\frac{\pi}{2} + \delta, \tag{11}$$

where p stands for the odd numbers 1, 3, 5 . . . while δ is a quantity which tends to zero and is the less the higher the number p. From (11) follows

$$x = p\cdot\frac{\lambda}{4} + \frac{\delta}{\pi}\cdot\frac{\lambda}{2}, \tag{12}$$

while the zero-points in the case considered are given by

$$x = p'\cdot\frac{\lambda}{2}, \tag{13}$$

where p' indicates the numbers 0, 1, 2, 3 . . .

The expression (2) may, as indicated above, be considered as describing the motion of the point of intersection between the wave and a plane at a distance x from the field and perpendicular to the axis of the original jet. It is seen that the said point performs harmonic vibrations. The zero-point of the latter is the point of intersection with the original jet. The vibrations are of course synchronous with the alternating current from which the wave originates. But they are delayed in phase with regard to the current, and the phase-lagging is

$$\varphi = \omega\frac{x}{v} = 2\pi\cdot\frac{x}{\lambda}. \tag{14}$$

From (14) it is seen that the vibrating point of intersection will pass the zero-point simultaneously with the current at a series of distances of the plane given by

$$2\pi\cdot\frac{x}{\lambda} = p\cdot\pi \tag{15}$$

or by

$$(16) \qquad x = p \cdot \frac{\lambda}{2},$$

where p indicates the numbers 1, 2, 3, 4 . . . The points determined by (16) may be called the nodes. With a short field they thus form a series of points separated by the constant distance $\frac{\lambda}{2}$. Obviously, if the wave is produced by a current $i = I \sin \omega t$, being zero at the moment $t = 0$, the nodes are simply determined by the points of intersection between the x-axis and the wave at the moment $t = 0$.

It should be noted that the wave considered in the present paragraph might as well have been produced by intersection of an alternating field and a constant current flowing in the jet.

On the vibratory motion of the hitting point of the jet-wave in a plane perpendicular to the axis of the original jet, and on the easily adjustable phase-displacement between the said motion and the current by which the wave is produced, the jet-wave commutators and rectifiers are based. The commutator generally serves for commutation of a voltage synchronous with the said current. The commutation is, as a rule, to take place nearly at the moment at which the voltage changes its sign. This is achieved by moving the commutator-electrode, consisting of two insulated parts symmetrically placed with regard to the axis of the jet-wave, in the direction of the said axis.

4. Geometrical Construction of the Jet-Wave.

From the expressions (2) and (4) in the previous paragraph it is seen that the angle θ which the path of the consecutive jet-particles forms with the axis of the original jet varies periodically with time according to the expression

$$(1) \qquad tg\,\theta = tg\,\theta_m \cdot \sin \omega t$$

a suitable zero-point for the time being assumed. It is therefore a simple matter to construct the paths of a series of consecutive particles. In fig. 5 this has been done in an easily comprehensible way for 16 particles following each other with a time-difference $\frac{T}{16}$. The paths are marked 0, 1, 2 ... 16. Now let the first particle 0 have reached the

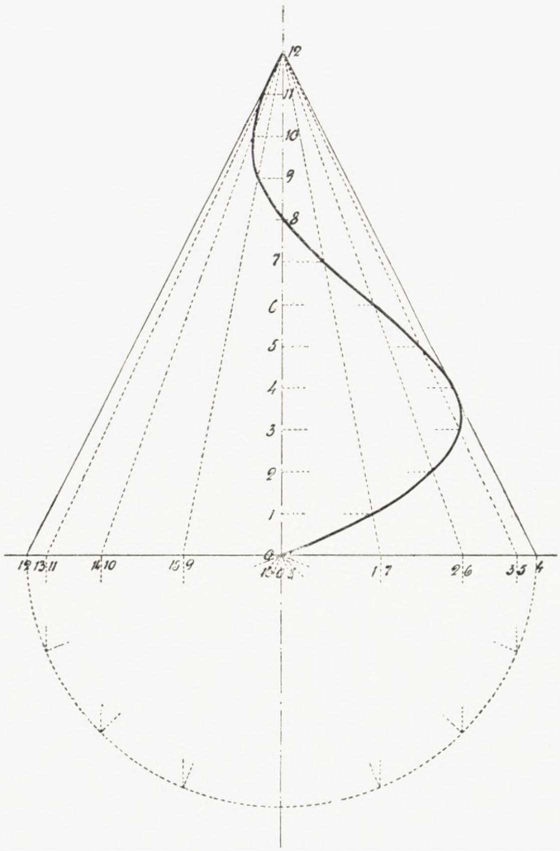


Fig. 5. Construction of Jet-Wave.

point 0 of the axis at a given moment. The next particle 1 will then be $\frac{\lambda}{16}$ behind the particle 0 on track 1, the particle 2 again $\frac{\lambda}{16}$ behind particle 1 on path 2 and so on. In this way a series of points of the wave at the moment considered is found, and the wave itself is easily traced. The picture in fig. 5 does not, however, give a true conception of an electromagnetically produced wave with an amplitude of the size shown. For the theory on which the construction is based holds good only for quite small amplitudes. The figure is therefore merely to be taken as illustrative of the way of constructing waves with small amplitudes.

5. The Jet-Wave in the Case of alternating Current and alternating Field.

If both current and field vary periodically say according to

$$i = I \sin \omega t \tag{1}$$

$$H = H_0 \sin (\omega t + \varphi) \tag{2}$$

it is found from the general theory in paragraph 2 that the wave produced may be represented by

$$\begin{aligned} y = & \frac{2}{10} \frac{I H_0}{m v^2} \cdot dl \cdot x \cdot \cos \varphi \\ & + \frac{1}{20} \frac{I H_0}{m v^2} \cdot dl \cdot x \cdot \sin \left[2 \omega \left(t - \frac{x}{v} \right) + \varphi - \frac{\pi}{2} \right]. \end{aligned} \tag{3}$$

From this expression it is seen that a wave is formed which has half the wave-length of that produced by a constant field. The wave proceeds in a direction which forms an angle θ_0 with the direction of the original jet determined by

$$tg\, \theta_0 = \frac{2}{10} \frac{I H_0}{m v^2} \cdot dl \cdot \cos \varphi . \tag{4}$$

The angle is thus zero for $\varphi = \frac{\pi}{2}$. In that case the wave becomes

$$y = \frac{1}{20} \frac{I H_0}{m v^2} dl \cdot x \cdot \sin 2 \omega \left(t - \frac{x}{v} \right) \tag{5}$$

and moves on in the direction of the original jet.

It is seen that the deflection θ_0 or rather $tg\, \theta_0$ might be taken as a measure of $\cos\varphi$. An indicator for phase-displacement might obviously be based on the relation (4).

6. Jet-Wave produced by a direct Current and a rotating Field.

Finally we shall consider the wave produced if a direct current I_0 is sent throught the jet while the latter with the original direction x, fig. 6, passes a rotating field H_0 of the period T, thus of the cyclic frequency $\omega = \frac{2\pi}{T}$. We may assume that direction of rotation as positive which seen against the positive direction of the x-axis coincides with that of a watch. With a field rotating in a positive direction the mechanical motive force $F = \frac{1}{10} I_0 H_0$ acting on one cm of the jet is delayed with regard to the field-vector by the angle $\frac{\pi}{2}$.

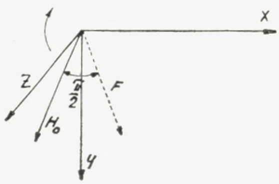


Fig. 6. Theory of Wave produced by direct Current in Interaction with a rotating Field.

The field may be produced by means of two periodic fields in the direction of the y-axis and z-axis respectively. The two component fields may be represented by

$$H_y = H_0 \cdot \cos \omega t \tag{1}$$

$$H_z = H_0 \sin \omega t. \tag{2}$$

Each field produces its motion and the actual motion is the resultant of the two. The problem has thus in a way already been solved. The two component waves are

$$y = \frac{1}{10} \frac{I_0 H_0}{m\, v^2} \cdot dl \cdot x \cdot \sin \omega \left(t - \frac{x}{v}\right) \tag{3}$$

$$z = \frac{1}{10} \frac{I_0 H_0}{m\, v^2} \cdot dl \cdot x \cdot \cos \omega \left(t - \frac{x}{v}\right). \tag{4}$$

The resultant wave, obviously, has the shape of a screw-line with a radius increasing in proportion to the distance

x from the field. Its point of intersection with a plane perpendicular to the x-axis traces a circle with the radius

$$(5) \qquad r = \frac{1}{10}\frac{I_0 H_0}{m\,v^2}\cdot dl\cdot x.$$

The radius to the point of intersection is delayed with regard to the motive force by the angle

$$(6) \qquad \varphi = \frac{x}{\lambda}\cdot 2\pi.$$

On the motion here considered the rotating jet-wave commutator is based.

7. Jet-Wave of small Amplitude with non-laminar Field.

We now proceed to consider the wave with a field of an extension L which is not small compared to the wave-length λ. The amplitude again is assumed to be relatively small as in the previous cases and again a current $i = I \sin \omega t$ is sent through the jet. One way of treating the problem is to divide the field into laminae dl as indicated in fig. 7 and to sum up the deviations Δy to which the said laminae give rise. The total deflection at a distance x from the entrance of the field and inside the same, $x < L$, is thus, according to the theory given above, determined by

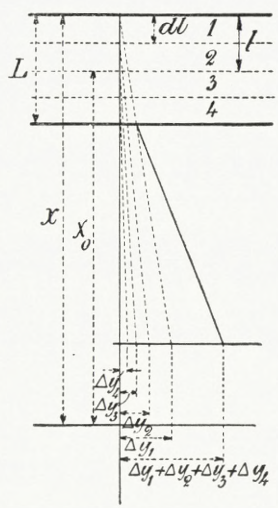


Fig. 7. Theory of Wave with non-laminar Field.

$$(1) \qquad y = \frac{1}{10}\frac{IH}{m\,v^2}\int_0^x (x-l)\sin\omega\left(t-\frac{x-l}{v}\right)dl.$$

Outside the field the deviation is given by

$$(2) \qquad y = \frac{1}{10}\,\frac{IH}{m\,v^2}\int_0^L (x - l)\sin\omega\left(t - \frac{x - l}{v}\right)dl.$$

In both cases l indicates the distance from the entrance of the field, which is assumed homogeneous, to the arbitrary lamina dl.

8. The Jet-Wave inside the Field.

From (1) in paragraph 7 is found by integration

$$(1) \qquad y = \frac{1}{10}\,\frac{IH}{m\,v^2}\sqrt{A^2 + B^2}\sin(\omega t - \varphi)$$

where

$$(2) \qquad A = \frac{\omega x}{v}\sin\frac{\omega x}{v} + \cos\frac{\omega x}{v} - 1$$

$$(3) \qquad B = \frac{\omega x}{v}\cos\frac{\omega x}{v} - \sin\frac{\omega x}{v}$$

$$(4) \qquad tg\,\varphi = -\frac{B}{A}.$$

Again the point of intersection of the wave with a plane perpendicular to the original jet will vibrate synchronously with the alternating current but lagging in phase with regard to the same. The point will pass the zero-point simultaneously with the current at a series of positions x of the said plane, the nodes, represented by

$$(5) \qquad tg\frac{\omega x}{v} = \frac{\omega x}{v}.$$

The nodes are, thus, no longer equidistant. Practically, however, they become so at greater distances from the entrance to the field, as will be seen from the well-known graphical way of solving (5). The final distance between two nodes, is, as with a short field,

$$(6) \qquad \frac{\lambda}{2} = v\frac{T}{2}.$$

The amplitude-curves, that is to say, the curves inside which the wave proceeds, are given by

$$(7)\quad \begin{aligned} y &= \pm\frac{1}{10}\,\frac{IH}{m\,v^2}\cdot\sqrt{A^2+B^2} = \\ &\pm\frac{1}{10}\,\frac{IH}{m\,\omega^2}\sqrt{(\beta'-\sin\beta')^2+(1-\cos\beta')^2}, \end{aligned}$$

β' being introduced for $\frac{\omega x}{v}$. Obviously with increasing distance x the curves approach the two straight lines

$$(8)\qquad y = \pm\frac{1}{10}\,\frac{IH}{m\,\omega^2}\cdot\beta'.$$

A good approximation is

$$(9)\qquad y = \pm\frac{1}{10}\,\frac{IH}{m\,\omega^2}(\beta'-\sin\beta').$$

The curve $y=\beta'-\sin\beta'$ is shown in fig. 8. It forms a kind of staircase profile ascending along the line $y=\frac{1}{10}\,\frac{IH}{m\,\omega^2}\beta'$, and having horizontal tangents at the equidistant points

(10) $\beta' = 0, 2\pi, 4\pi \ldots$

or at distances from the entrance to the field given by

(11) $x = 0, \lambda, 2\lambda \ldots$

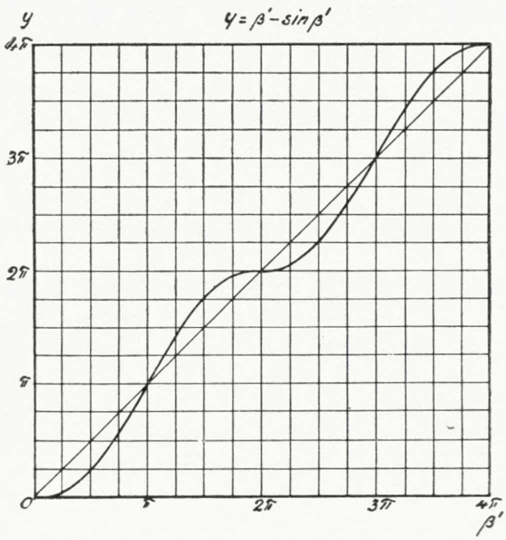


Fig. 8. Amplitude-Curve for Wave inside the Field.

The same is true for the curve represented by the exact expression (7). We may conclude that if the field is cut off at one of the distances given by (11), we shall obtain a wave with a constant amplitude outside the field.

9. The Jet-Wave outside the Field.

For the jet-wave outside the field the following expression is found by integration of (2) paragraph 7:

$$(1)\quad \begin{aligned} y = \frac{2}{10}\frac{IH}{m\omega^2}\Bigg[&\left(\frac{\omega\frac{L}{2}}{v}\cos\frac{\omega\frac{L}{2}}{v}-\sin\frac{\omega\frac{L}{2}}{v}\right)\cos\omega\left(t-\frac{x-\frac{L}{2}}{v}\right) \\ &+\frac{\omega\left(x-\frac{L}{2}\right)}{v}\sin\frac{\omega\frac{L}{2}}{v}\sin\omega\left(t-\frac{x-\frac{L}{2}}{v}\right)\Bigg]. \end{aligned}$$

For $x = L$ this expression gives the same value for y as does the formula (7) paragraph 8 for the wave inside the field. It shows that the motion of the point of intersection with a plane perpendicular to the original jet is synchronous with the current by which the wave is produced. The phase-lagging, however, is now given by

$$(2)\quad \varphi = \varphi' - \frac{\omega\left(x-\frac{L}{2}\right)}{v},$$

where

$$(3)\quad tg\,\varphi' = \frac{\frac{\omega\frac{L}{2}}{v}\cos\frac{\omega\frac{L}{2}}{v}-\sin\frac{\omega\frac{L}{2}}{v}}{\frac{\omega\left(x-\frac{L}{2}\right)}{v}\sin\frac{\omega\frac{L}{2}}{v}}$$

or

$$(4)\quad tg\,\varphi' = \frac{\gamma\frac{\pi}{2}\cos\gamma\frac{\pi}{2}-\sin\gamma\frac{\pi}{2}}{\beta\pi\sin\gamma\frac{\pi}{2}},$$

if the length of the field measured with $\frac{\lambda}{2}$ as unit is indicated by γ and if the distance from the centre of the field to the plane measured in the same unit, thus $\dfrac{x-\dfrac{L}{2}}{\dfrac{\lambda}{2}}$, is denoted by β. We shall now discuss the wave considered in detail.

10. Amplitude of Wave outside the Field.

As will be seen, the amplitude-curve may be expressed by

$$(1) \quad y=\frac{2}{10}\,\frac{IH}{m\,\omega^2}\cdot\sqrt{\left(\gamma\frac{\pi}{2}\cos\gamma\frac{\pi}{2}-\sin\gamma\frac{\pi}{2}\right)^2+\left(\beta\pi\sin\gamma\frac{\pi}{2}\right)^2}.$$

With increasing distance β from the centre of the field (1) tends to

$$(2) \quad y=\frac{2}{10}\cdot\frac{IH}{m\,\omega^2}\,\beta\cdot\pi\cdot\sin\gamma\frac{\pi}{2}$$

or to

$$(3) \quad y=\frac{1}{10}\,\frac{IH}{m\,v^2}\cdot L\cdot x_0\,\frac{\sin\gamma\frac{\pi}{2}}{\gamma\frac{\pi}{2}},$$

x_0 standing for $x-\dfrac{L}{2}$.

This expression may be compared with the formula for the amplitude with a short field, thus with (3) paragraph 3. The two expressions are identical apart from the factor $\dfrac{\sin\gamma\frac{\pi}{2}}{\gamma\frac{\pi}{2}}$ provided the distances, in the case of the non-laminar field, are measured from the centre of the field. Thus at greater distances the formula corresponding to a short field may be used provided the result is reduced by applying the factor

$$(4) \quad F=\frac{\sin\gamma\frac{\pi}{2}}{\gamma\frac{\pi}{2}}.$$

With special extensions of the field particular waves are obtained. If for instance γ assumes such values that the first member under the square root in (1) vanishes, that is to say, if

$$tg\,\gamma\frac{\pi}{2} = \gamma\frac{\pi}{2}, \tag{5}$$

then (2) is exactly true for all distances β and the phase-displacement φ', (3) paragraph 9, also vanishes so that the whole phase-lagging is that expressed by $\frac{\omega\left(x-\frac{L}{2}\right)}{v}$. In the said case, therefore, the wave is identical with the wave with a short field, the length dl of which is determined by

$$dl = L\frac{\sin\gamma\frac{\pi}{2}}{\gamma\frac{\pi}{2}}. \tag{6}$$

The first length of the field satisfying (5) is obviously determined by a value of $\gamma\frac{\pi}{2}$ somewhat smaller than $3\frac{\pi}{2}$ thus by γ a little smaller than 3.

If on the other hand γ has such a value that

$$\sin\gamma\frac{\pi}{2} = 0 \tag{7}$$

thus if γ is equal to 2, 4, 6 etc. then

$$y = \frac{1}{10}\,\frac{IH}{m\,v^2}\cdot\frac{\gamma}{\pi}\cdot\left(\frac{\lambda}{2}\right)^2. \tag{8}$$

That is to say, the amplitude is constant outside the field as predicted in paragraph 8. At the same time $tg\,\varphi' = \infty$ or $\varphi' = p\cdot\frac{\pi}{2}\cdot(p = 1, 3, 5\ldots)$ and the formula for the wave becomes

$$(9) \qquad y = \frac{1}{10} \frac{IH}{m\,v^2} \cdot \frac{\gamma}{\pi} \cdot \left(\frac{\lambda}{2}\right)^2 \cos \omega \left(t - \frac{x_0}{v}\right).$$

A wave with constant amplitude may thus be electromagnetically produced, but a rather long field must be employed. It is easy to see how the amplitude becomes constant. The shortest field giving the wave in question is just a wave-length λ. It thus takes a particle a period to pass the field. The velocity in the positive direction of the y-axis which the particle obtains during the one half of the passage is therefore lost during the other. So the particle arrives at the boundary of the field with no velocity perpendicular to the direction of the original jet. But obviously it arrives with a certain deviation, the deviation due to the first half of the field not being compensated by the opposite deviation to which the second half gives rise, simply because the latter part of the field is closer to the exit of the field than the former.

11. The Variation of the Amplitude with the Extension of the Field at a given Distance from the Centre of the Field.

In addition to the general discussion in the preceding paragraph we may consider the variation of the amplitude with the extension of the field at a given distance from the centre of the field. We may in (1) paragraph 10 write

$$(1) \qquad \gamma \frac{\pi}{2} = \gamma' \quad \text{and} \quad (2) \quad \beta \pi = \beta'$$

with which notations the amplitude is

$$(3) \qquad y = \frac{2}{10} \frac{IH}{m\,\omega^2} \cdot \sqrt{(\gamma' \cos \gamma' - \sin \gamma')^2 + (\beta' \sin \gamma')^2}.$$

We find that y is minimum for the values of γ' given by

$$\sin \gamma' = 0 \tag{4}$$

or for

$$L = p \cdot \lambda \; (p = 1, 2, 3 \ldots) \tag{5}$$

thus for the values of L giving waves of constant amplitude, the latter, as shown, being expressed by

$$y = \frac{1}{10} \frac{IH}{m v^2} \cdot \frac{\gamma}{\pi} \cdot \left(\frac{\lambda}{2}\right)^2. \tag{6}$$

The amplitude is maximum for the field-extensions determined by

$$(\beta'^2 - \gamma'^2) \cos \gamma' + \gamma' \sin \gamma' = 0 \tag{7}$$

thus for

$$tg\, \gamma' = \frac{\gamma'^2 - \beta'^2}{\gamma'}. \tag{8}$$

The equation (8) may be solved graphically by determining the points of intersection between the two curves

$$y' = tg\, \gamma' \tag{9}$$

and

$$y = \frac{\gamma'^2 - \beta'^2}{\gamma'} = \gamma' \left(1 - \left(\frac{\beta'}{\gamma'}\right)^2\right). \tag{10}$$

If for instance $\beta' = \pi \beta = 3\pi$, corresponding to the amplitude being considered at a distance $\frac{3\lambda}{2}$ from the centre of the field, it is found that the amplitude is maximum for values of γ' close to $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ thus for extensions L nearly equal to $\frac{\lambda}{2}$ and $\frac{3\lambda}{2}$. In both cases the amplitude is about the same, namely approximately

$$y = \frac{2}{10} \frac{IH}{m \omega^2} \cdot \sqrt{1 + \beta'^2}. \tag{11}$$

thus for $\beta' = 3\pi$ very nearly

$$y = \frac{2}{10} \frac{IH}{m \omega^2} \cdot 3\pi = \frac{1}{10} \frac{IH}{m v^2} \cdot \frac{3}{2\pi} \cdot \lambda^2. \tag{12}$$

Finally the amplitude in the case considered is maximum for a value of γ' somewhat greater than $\frac{5\pi}{2}$. This is seen from fig. 9 representing the graphical solution of (8) with regard to the root considered. It is seen that the solution is $\gamma' = 20.95 \cdot \frac{\pi}{2}$ corresponding to $L = 5.24 \frac{\pi}{2}$.

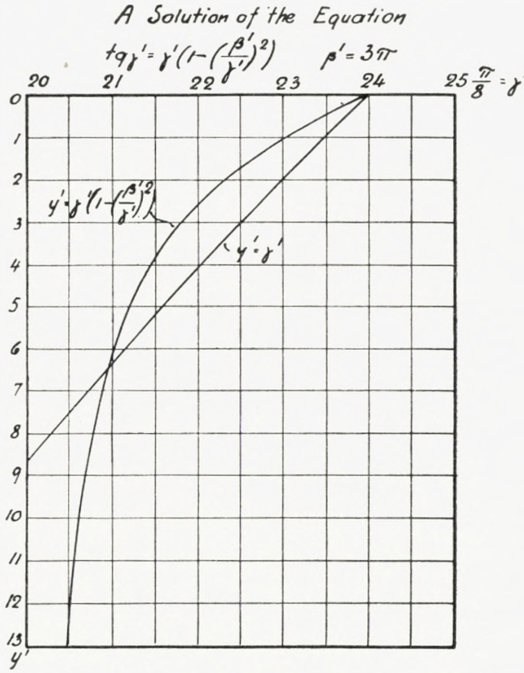


Fig. 9. Graphical Solution of Maximum-Problem.

In fig. 11 a complete graphical representation of the variation of the amplitude with γ' is given for $\beta' = 3\pi$ ($\beta = 3$). All three maximum-values of the amplitude are very nearly identical. The value of the minimum-amplitude increases according to a straight line. From these facts it follows that the difference between maximum and minimum becomes the less the greater the extension of

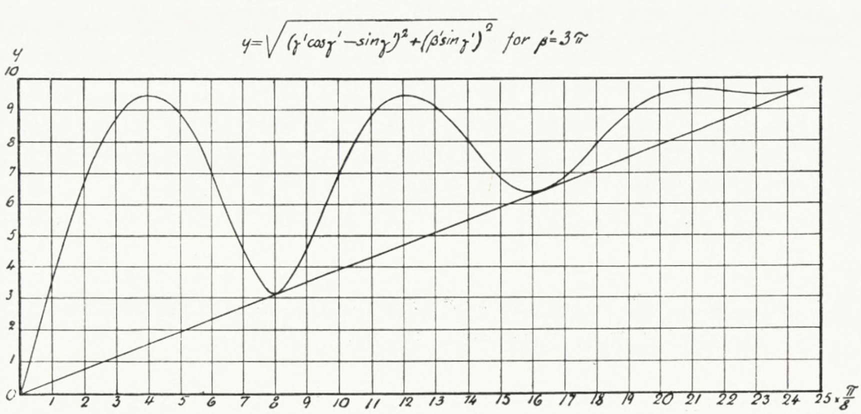


Fig. 10. Variation of Amplitude with Field-Extension, $\beta' = 3\pi$.

the field. Furthermore the conclusion may be drawn from fig. 10 that if the greatest possible amplitude with the cheapest possible magnet is aimed at, an extension of the field equal to $\frac{\lambda}{2}\left(\gamma' = \frac{\pi}{2}\right)$ should be chosen.

12. Position of the Nodes with Fields of various Extensions.

We may finally discuss the positions of the nodes in their dependency of the extension of the field. According to paragraph 9, (2) the nodes are determined by

$$\varphi = \varphi' - \beta' = p \cdot \pi \, (p = 0, 1, 2, 3, \ldots), \tag{1}$$

where with the notation in the previous paragraph

$$tg\, \varphi' = \frac{\gamma' \cos\gamma' - \sin\gamma'}{\beta' \sin\gamma'}. \tag{2}$$

The nodes are thus determined by

$$tg\, \beta' = \frac{\gamma' \cos\gamma' - \sin\gamma'}{\beta' \sin\gamma'}. \tag{3}$$

The values of β' satisfying (3) are found graphically as points of intersection between

$$\text{(4)} \qquad y' = \frac{c}{\beta'} \quad \text{and} \quad \text{(3)} \quad y' = tg\, \beta',$$

where

$$c = \frac{\gamma' \cos\gamma' - \sin\gamma'}{\sin\gamma'}.$$

In fig. 11 the solution is given for a series of values of $\gamma' = \frac{\pi}{2} \cdot \frac{L}{\frac{\lambda}{2}}$. For $\gamma' = 0\,(L = 0)$, c assumes the form $\frac{0}{0}$. The value is, however, easily seen to be 0. The nodes are thus determined as the points of intersection between the axis of abscissae and the tangent-curves or by $\beta' = \pi\beta = p\pi$, $(p = 1, 2, 3, \ldots)$ corresponding to distances from the field equal to $\frac{\lambda}{2}$, $\frac{2\lambda}{2}$, $\frac{3\lambda}{2}$ and so on, as already found in the discussion of the wave with a short field. If now the length

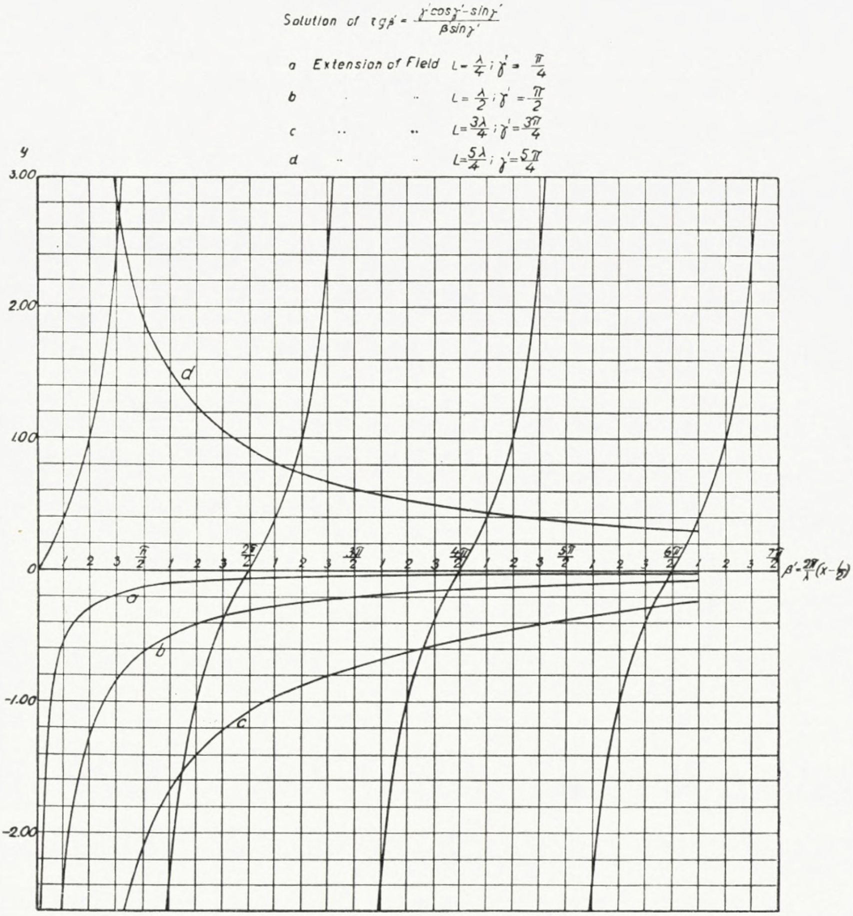


Fig. 11. Graphical Determination of the Nodes.

of the field is increased, c will become, and remain, negative and numerically increase steadily as long as

$$\gamma' < \pi$$

that is to say as long as

$$L < \lambda.$$

The positions of the nodes are determined by the points of intersection between the tangent-curves and hyperbolas situated below the positive part of the axis of abscissae.

In fig. 11 the hyperbolas are drawn corresponding to $\gamma' = \frac{\pi}{4}$ (a), $\gamma' = \frac{\pi}{2}$ (b), and $\gamma' = 3\frac{\pi}{4}$ (c) or to L equal to $\frac{\lambda}{4}, \frac{\lambda}{2}$ and $\frac{3\lambda}{4}$ respectively. It should be noted that the distances of the nodes from the centre of the field are approximately the same as with a short field all up to $L = \frac{3\lambda}{4}$ or even above. This is especially true for the nodes at greater distances from the field. For $\gamma' = \pi\,(L = \lambda)$, $tg\,\varphi'$ is ∞ thus $\varphi' = (2p_1 + 1)\cdot\frac{\pi}{2}$ where p_1 stands for 1, 2, 3 . . . The positions of the nodes are now determined by

$$\varphi = (2p_1 + 1)\cdot\frac{\pi}{2} - \beta' = p\pi, \tag{6}$$

from which

$$\beta = \frac{2\,(p_1 - p) + 1}{2} = \frac{2p_2 + 1}{2} = \frac{\beta'}{\pi}, \tag{7}$$

where p_2 stands for 1, 2, 3 . . . The nodes are thus all displaced by $\frac{\lambda}{4}$ with regard to the positions with a short field. This is also seen from fig. 11 if we imagine the hyperbolas extended infinitely. Now if γ' is increased beyond π (L beyond λ), c becomes positive and the hyperbola $y' = \frac{c}{\beta'}$, is situated above the axis of abscissae. For $\gamma' = \frac{5}{4}\pi\left(L = \frac{5}{4}\lambda\right)$ the hyperbola is d fig. 11. Again the nodes are distributed approximately as with a short field. The constant c remains positive until the first of the field-extensions (greater than 0) which satisfy

$$tg\,\gamma' = \gamma' \tag{8}$$

is reached. At the said extension L, which is a little less than $\frac{3}{2}\lambda$, the nodes are distributed exactly as in the case of a short field. If the extension is made still greater the constant c again becomes negative and so on.

In tab. I which is derived from fig. 11 the distances β_1, β_2, β_3 from the centre of the field of the three first nodes are given corresponding to four values of the field-extension.

Table I.

L	β_1	β_2	β_3	$\beta_2 - \beta_1$	$\beta_3 - \beta_2$
0	1.000	2.000	3.000	1.000	1.000
$\frac{\lambda}{4}$	0.981	1.988	2.994	1.007	1.006
$\frac{\lambda}{2}$	0.894	1.950	2.975	1.056	1.025
$\frac{3\lambda}{4}$	0.695	1.830	2.918	1.135	1.088

It is seen from the last two additional columns how nearly equal to $\frac{\lambda}{2}$ the distance between consecutive nodes is, even with fields of rather great extension.

13. The Jet-wave produced by an oscillating Nozzle.

A jet-wave of the kind considered above, but subject to no restrictive assumption with regard to the size of the amplitude, may be produced by means of an oscillating nozzle. In fig. 12 N indicates a nozzle which per-

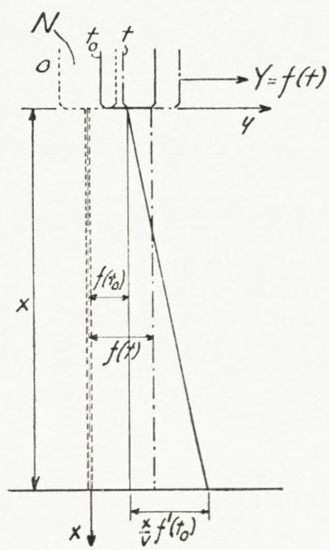


Fig. 12. Wave produced by an oscillating Nozzle.

forms a translatory motion perpendicular to the axis of the nozzle, the motion being determined by

$$Y = f(t). \tag{1}$$

A jet-particle which leaves the jet-hole at the moment t_0 will, at the distance x from the nozzle, exhibit a deviation with regard to the axis of the said nozzle determined, as seen from the figure, by

$$y = f(t_0) + \frac{x}{v} f'(t_0), \tag{2}$$

$f'(t_0)$ standing for the velocity $\frac{dy}{dt}$ at the moment t_0, and v for the velocity of the jet. If furthermore the distance x has been reached at the moment t, then

$$x = v(t - t_0). \tag{3}$$

The expression for the wave produced is found by eliminating t_0 from (2) and (3). It is

$$y = f\left(t - \frac{x}{v}\right) + \frac{x}{v} f'\left(t - \frac{x}{v}\right). \tag{4}$$

If now, particularly

$$Y = Y_0 \sin \omega t \tag{5}$$

then

$$\begin{aligned} y &= Y_0 \left[\sin \omega \left(t - \frac{x}{v}\right) + \frac{x}{v} \omega \cos \omega \left(t - \frac{x}{v}\right)\right] \\ &= Y_0 \sqrt{1 + \left(\frac{\omega x}{v}\right)^2} \sin(\omega t - \varphi), \end{aligned} \tag{6}$$

where

$$\varphi = \frac{\omega x}{v} - \operatorname{arc} tg \frac{\omega x}{v}. \tag{7}$$

The amplitude-curve of the wave is expressed by

$$y = Y_0 \sqrt{1 + \left(\frac{\omega x}{v}\right)^2} \tag{8}$$

or by

$$\frac{y^2}{Y_0^2} - \frac{x^2}{\left(\frac{v}{\omega}\right)^2} = 1 . \tag{9}$$

It is thus a hyperbola with the axis Y_0 and $\frac{v}{\omega} = \frac{\lambda}{2\pi}$ $\left(\lambda = vT,\ T = \frac{2\pi}{\omega}\right)$. At greater distances from the jet-hole the amplitude-curve may be represented by

$$y = \pm Y_0 \frac{x\,\omega}{v} = \pm Y_0 \frac{2\pi}{\lambda} \cdot x \tag{10}$$

thus by two straight lines.

The formula (6) may also be conceived as the equation of the motion of the point of intersection between the jet-wave and a plane perpendicular to the axis of the original jet. The motion is harmonic like that of the nozzle but lags in phase with regard to the latter. The phase-displacement is zero or π at the points at which the wave cuts the x-axis at the moment $t = 0$, that is to say, at the points determined by

$$tg\, \omega \frac{x}{v} = \omega \frac{x}{v} . \tag{11}$$

Attention may be drawn to the fact that the nodes inside the field of an electromagnetically produced wave were determined by just the same equation ((5) paragraph 8), x being the distance from the entrance to the field.

If the deviation of the particle is measured relatively to the axis of the moving nozzle it is expressed by

$$y = \frac{x}{v} f'(t_0) - \left[f(t) - f(t_0)\right] \tag{12}$$

(compare fig. 12). With harmonic oscillation of the nozzle the equation of the jet-wave in the oscillating system of coordinate is

$$y = Y_0\left[\left(\omega\frac{x}{v}\cos\omega\left(t-\frac{x}{v}\right)-\sin\omega t+\sin\omega\left(t-\frac{x}{v}\right)\right]\right.$$

$$(13)\qquad = Y_0\left[\left(\omega\frac{x}{v}\sin\omega\frac{x}{v}+\cos\omega\frac{x}{v}-1\right)\sin\omega t\right.$$

$$\left.+\left(\omega\frac{x}{v}\cos\omega\frac{x}{v}-\sin\omega\frac{x}{v}\right)\cos\omega t\right].$$

The wave is seen to be identical with an electromagnetically produced wave of small amplitude inside the field provided (paragraph 8)

$$(14)\qquad Y_0 = \frac{1}{10}\frac{IH}{m\omega^2}.$$

It has thus already been discussed above.

On the wave-motion considered in the present paragraph the jet-wave accelerometer is based[1]. If $f(t)$ in (12) is replaced by the first three terms of the series

$$(15)\qquad f(t) = f(t_0) + (t-t_0)f'(t_0) + \tfrac{1}{2}(t-t_0)^2 f''(t_0) + -$$

where $t - t_0 = \frac{x}{v}$, (12) may be written

$$(16)\qquad y = -\tfrac{1}{2}\left(\frac{x}{v}\right)^2 f''(t_0) = -\tfrac{1}{2}\left(\frac{x}{v}\right)^2 f''\left(t-\frac{x}{v}\right),$$

it being assumed that the displacement of the nozzle is small and that y is measured so close to the nozzle that the members of higher order of (15) may be neglected. It is thus seen that the acceleration $f''(t_0)$ of the nozzle or of any body to which the nozzle is attached may be registered by the relative motion of a point of the jet-wave close to the nozzle.

On the other hand it is seen that (12) at greater distances assumes the shape

$$(17)\qquad y = \frac{x}{v}f'(t_0)$$

provided again that the displacement of the nozzle is kept within certain limits. The velocity of a body to which the nozzle is attached is thus registered by the relative motion of a point of the jet-wave chosen not too close to the nozzle.

[1] Nature, June 6, 1925, and Phil. Mag., vol. III, 1073.

CHAPTER II

The Jet-Wave of large Amplitude.

1. The Jet-Wave in the Case of a laminar Field.

We now proceed to consider jet-waves of larger amplitudes and shall commence with a wave produced by a laminar field i. e. a field the extension dl of which is so small that the current used in the production of the wave may be considered constant during the passage of a small particle Δx of the jet. While, in the building up of the theory in the case of small amplitudes, we were justified in assuming the mechanical force, acting on the jet-particle, perpendicular to the axis of the original jet, this assumption can now no longer be maintained. During the passage the particle Δx of the jet will be attacked by a force

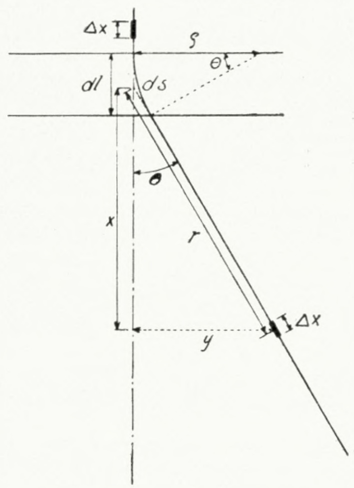


Fig. 13. Theory of Wave with short Field.

$$K = \frac{1}{10} iH\Delta x, \tag{1}$$

where i is the value of the current during the passage and H is the intensity of the homogeneous field. The force K

is perpendicular to the field and to the direction of Δx or, what we shall assume to be the same, to the direction of motion of the said particle. It is therefore unable to alter the original velocity v of the particle, but it will force the latter to follow a circular path the radius ϱ of which is determined by

$$\frac{m \cdot \Delta x \cdot v^2}{\varrho} = K, \tag{2}$$

m being as in Chapt. I the mass per cm of the original jet. From (1) and (2) we derive

$$\frac{1}{\varrho} = \frac{1}{10} \cdot \frac{IH}{m v^2}. \tag{3}$$

After having left the field the particle will proceed along a straight line forming an angle θ with the direction of the original jet, where obviously

$$\sin \theta = \frac{dl}{\varrho} = \frac{1}{10} \cdot \frac{IH}{m v^2} \cdot dl. \tag{4}$$

If the current through the jet is determined by

$$i = I \sin \omega t \tag{5}$$

and if the particle which passes the field at the moment t_0 is considered, the deflection of the path is expressed by

$$\sin \theta = \frac{1}{10} \cdot \frac{IH}{m v^2} \cdot dl \cdot \sin \omega t_0. \tag{6}$$

At a later moment t the particle will have reached a distance r from the field, where

$$r = v(t - t_0). \tag{7}$$

Eliminating t_0 from (6) and (7) we find the formula of the jet-wave. It is

$$\text{(8)} \quad \begin{aligned} \sin\theta &= \frac{1}{10}\cdot\frac{IH}{m\,v^2}\cdot dl\cdot\sin\omega\left(t-\frac{r}{v}\right) \\ &= \sin\theta_m\cdot\sin\omega\left(t-\frac{r}{v}\right). \end{aligned}$$

Obviously (8) represents a wave proceeding between two straight lines $\theta = \pm\theta_m$ where θ_m is seen to be the stationary angle of deflection for a jet carrying a constant current I. The wave-length is determined by

$$\text{(9)} \qquad \lambda = v\cdot T$$

as in the case of a wave of small amplitude. The equation derived for the latter wave was

$$\text{(10)} \quad tg\,\theta = \frac{y}{x} = \frac{1}{10}\cdot\frac{IH}{m\,v^2}\cdot dl\cdot\sin\omega\left(t-\frac{x}{v}\right) = tg\,\theta_m\cdot\sin\omega\left(t-\frac{x}{v}\right).$$

The latter formula may be used instead of (8) for the determination of the amplitude θ_m as long as the difference between $\sin\theta_m$ and $tg\,\theta_m$ may be neglected. For $\theta_m = 10^\circ$ $\sin\theta_m = 0.1736$ and $tg\,\theta_m = 0.1763$. The difference in this case is 0.0027 or ab. 1.5 per cent. For $\theta_m = 20^\circ$ the difference is already 6.3 per cent.

2. Construction of the Jet-Wave.

Fig. 14 illustrates how a jet-wave of given angular amplitude θ_m is constructed. The angle θ_m is laid down to each side of the direction of the original jet *OP*. With an arbitrary part *OA* of the line *OP* as diameter a circle is drawn. It cuts the line *OC* limiting the track of the wave in the point *B*. The chord *AB* is swung down round *A* on the line *CC* perpendicular to *OA*. In this way the end of the chord *AB* comes down at *B'*. We may now for instance consider 17 consecutive particles of the jet following each other at a mutual distance of $\frac{\lambda}{16}$. In order to determine the rectilinear paths of the said particles, 17 pieces are set out on the line $B'B'$ from the point A. The pieces are as $\sin 0\cdot\frac{\pi}{8}$, $\sin 1\cdot\frac{\pi}{8}$, $\sin 2\cdot\frac{\pi}{8}$, ..., $\sin 15\cdot\frac{\pi}{8}$, $\sin 16\cdot\frac{\pi}{8}$.

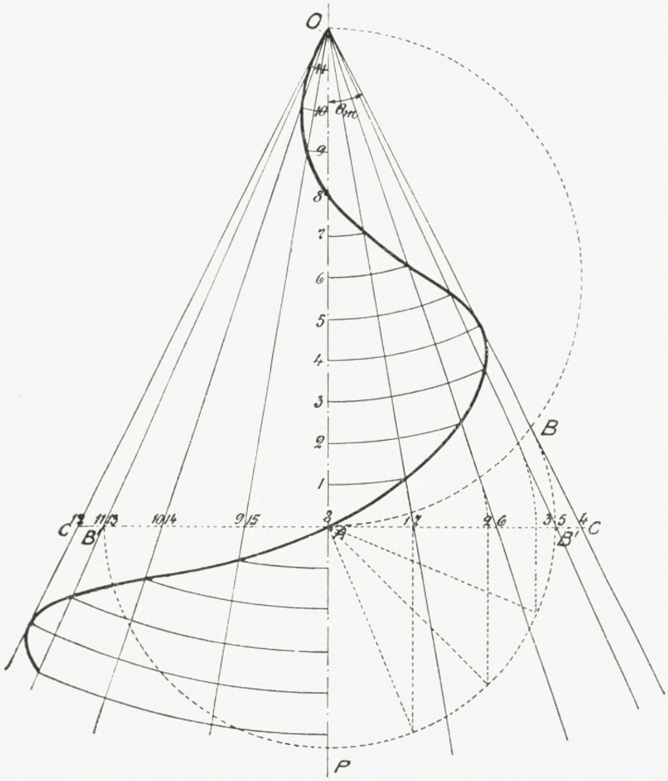


Fig. 14. Construction of Jet-Wave.

AB' is taken to represent the unit length. The well-known construction of the pieces is indicated in the figure. Now the ends of the same pieces (0), 1, 2, 3, ..., 15, (16) are swung back round A on the circle OBA. Through the points thus obtained the tracks of the particles in question pass. They may be numbered 0, 1, 2, ..., 15, 16. Now assuming that at a given moment particle 0 has arrived at A

on the track OA, then particle 1 will be on track 1 nearer to the point O by the distance $\frac{\lambda}{16}$, particle 2 will be on track 2 nearer by $\frac{\lambda}{16}$ to O than particle 1 etc. The way

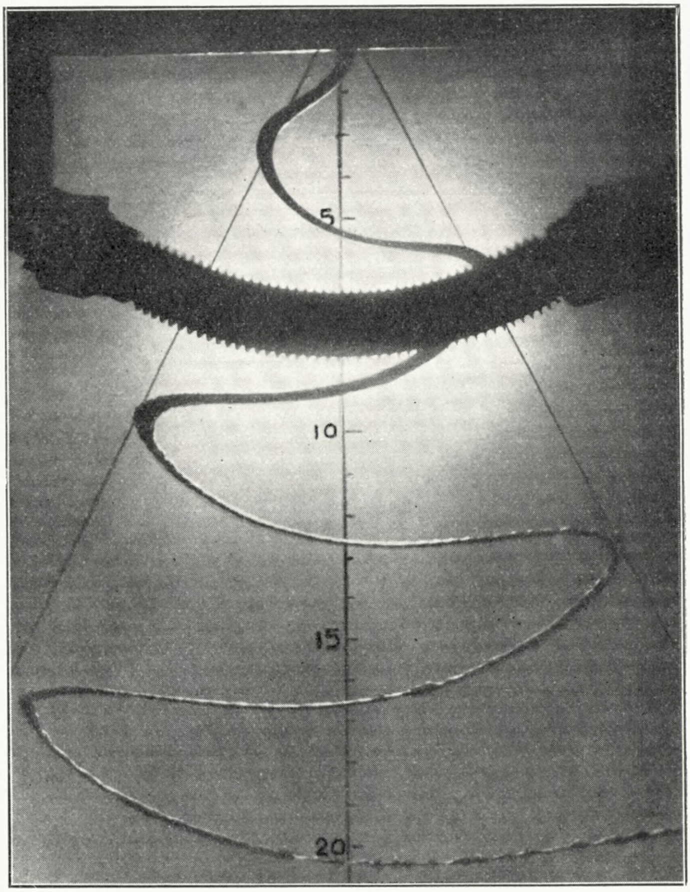


Fig. 15. Jet-Wave, phot.

to find the positions of the consecutive particles is thus obvious. Backwards from A we may divide OA in parts, each of the length $\frac{\lambda}{16}$, and mark the points of division by 1, 2, 3, ... Then in order to find the position of a certain particle, we shall only have to project the point of division of the corresponding number on to the track of the same number by means of a circle with its centre at O.

A main difference between the construction of the wave with small amplitude (or of the wave produced by means of a nozzle performing translatory oscillations) and that of the wave with large amplitude lies in the way of projecting the points of division of the jet-axis on to the tracks. In the first case the projection takes place by lines at right angles with the said axis, in the second by circles. The first type of waves may accordingly be characterized as the rectangular type while the second may be termed the circular type. The latter type has the peculiarity of the wave-fronts being markedly convex in the direction of the motion. How close the actual jet-wave comes to the shape of the constructed wave is seen from fig. 15, representing an instantaneous photograph of a wave produced electromagnetically from a mercury-jet.

3. The Jet-Wave in the Case of a non-laminar Field. General Theory.

We now proceed to consider the wave with a field which is not laminar. Again we shall assume the wave to be produced by the current $i = I \sin \omega t$ in interaction with a constant and homogeneous field. The extension of the latter in the direction of the original jet is L, fig. 16.

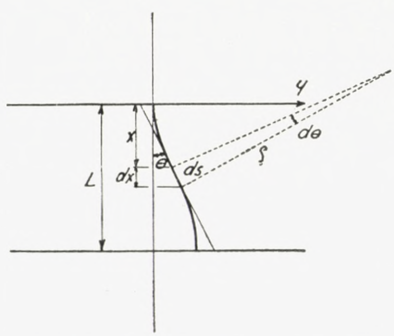


Fig. 16. Theory of Wave with non-laminar Field.

During the motion through the lamina dx of the field the path of the particle considered will suffer a change of direction $d\theta$, given, as will be seen from the figure, by

$$d\theta = \frac{ds}{\varrho} = \frac{v\,dt}{\varrho}, \tag{1}$$

where

$$\frac{m\,v^2}{\varrho} = \frac{1}{10}\cdot HI \sin \omega t, \tag{2}$$

m indicating as above the mass per cm of the original jet. From (1) and (2) is found

$$d\theta = \frac{1}{10}\,\frac{HI}{m\,v^2}\cdot v\cdot \sin \omega t \cdot dt, \tag{3}$$

and by integration

$$\begin{aligned} \theta &= \frac{1}{10}\cdot\frac{HI}{m\,v^2}\cdot\frac{\lambda}{2\pi}(\cos \omega t_0 - \cos \omega t) \\ &= \theta_0 (\cos \omega t_0 - \cos \omega t), \end{aligned} \tag{4}$$

t_0 being the moment at which the particle enters the field. The equation (4) gives the direction of the motion of a particle at any point of the path provided it is known at what moment t the particle is at the point in question. Especially it is possible to determine the direction of the motion at the lower boundary of the field if the time of passage of the field is known. We shall now derive a formula expressing the distance x from the upper boundary of the field as a function of time.

From fig. 16 it is seen that

$$dx = ds\cdot \cos\theta = v\,dt \cos\theta. \tag{5}$$

If θ is kept below a certain limit we may put

$$\cos\theta = 1 - \frac{\theta^2}{2}. \tag{6}$$

The error committed is of the order $\frac{\theta^4}{24}$ thus, with $\theta = \frac{1}{2}$, $\frac{1}{384}$ or 0.3 per cent.

If the value of $\cos\theta$ determined by (6) is introduced in (5) and if the value of θ is then taken from (4), an integration gives

$$\begin{aligned} x &= \left(1 - \frac{\theta_0^2}{4} - \frac{\theta_0^2}{2}\cos^2\omega t_0\right) v\,(t - t_0) \\ &\quad + v\frac{\theta_0^2}{\omega}\cos\omega t_0\,(\sin\omega t - \sin\omega t_0) \qquad (7) \\ &\quad - v\frac{\theta_0^2}{8\,\omega}(\sin 2\,\omega t - \sin 2\,\omega t_0) \end{aligned}$$

or

$$\begin{aligned} x &= \left(1 - \frac{\theta_0^2}{4} - \frac{\theta_0^2}{2}\cos^2\omega t_0\right)\lambda\cdot\frac{t - t_0}{T} \\ &\quad + \lambda\frac{\theta_0^2}{2\pi}\cos\omega t_0\,(\sin\omega t - \sin\omega t_0) \qquad (8) \\ &\quad - \lambda\frac{\theta_0^2}{16\,\pi}(\sin 2\,\omega t - \sin 2\,\omega t_0). \end{aligned}$$

Finally a formula for the deviation y perpendicular to the original jet of the particle is derived. From fig. 16 it appears that

$$dy = ds\cdot\sin\theta = v\,dt\cdot\sin\theta. \qquad (9)$$

An approximation which will suffice in most cases is obtained by replacing $\sin\theta$ by θ. The error is of the order $\frac{\theta^3}{6}$ i. e. ab. 2 per cent. for $\theta = \frac{1}{2}$. Apparently this error is not quite small but as y in most practical cases is small, the error is only of relatively small influence on the amplitude outside the field at greater distances from the same. Introducing the value of θ from (4) and integrating, we find

$$y = v\theta_0\,(t - t_0)\cdot\cos\omega t_0 - \frac{v\,\theta_0}{\omega}(\sin\omega t - \sin\omega t_0) \qquad (10)$$

or

$$(11) \qquad y = \lambda\,\theta_0 \frac{t - t_0}{T} \cdot \cos\omega\, t_0 - \lambda \frac{\theta_0}{2\pi} (\sin\omega\, t - \sin\omega\, t_0)\,.$$

For small values of θ_0 the theory now developed coincides with the theory for a wave of small amplitude in-

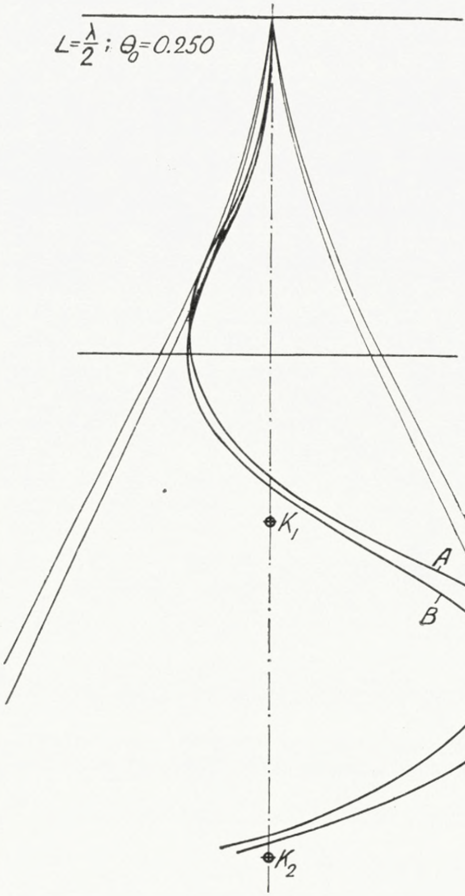


Fig. 17. Difference between Waves calculated from the exact Theory and from the Theory with small Amplitude.

side a field of great extension. If the members with θ_0^2 of (7) may be neglected, the formula is reduced to

$$(12) \qquad x = v\,(t - t_0)$$

and by eliminating t_0 from (11) and (12) we find for the equation of the wave

$$y = \frac{1}{10}\,\frac{IH}{m\,v^2}\left(\frac{\lambda}{2\pi}\right)^2\left[\frac{2\pi x}{\lambda}\cos\omega\left(t-\frac{x}{v}\right)-\sin\omega t + \sin\omega\left(t-\frac{x}{v}\right)\right], \tag{13}$$

which may be shown to be identical with the formula (1) in Chapt. I, paragraph 8. With large amplitudes the expression (13) leads to a false picture of the wave both inside and outside the field. Fig. 17 illustrates this. The wave A is calculated on the basis of the complete theory, while B is found by applying the theory for small amplitudes. In both cases the same value for $\frac{1}{10}\,\frac{IH}{m\,v^2}\cdot\frac{\lambda}{2\pi} = \theta_0$ is assumed. By employing the theory for small amplitudes a too small value of the amplitude is found, and at the same time the zero points or nodes are displaced somewhat with regard to the true nodes, i. e. the points of intersection between A and the axis of the wave. The points K_1 and K_2 represent the positions of the nodes as they would be with a laminar field in the centre of the actual field. How pictures like A in fig. 17 are produced, will now be explained.

4. Production of Wave-Pictures on the Basis of the complete Theory.

By means of the theory of paragraph 3 it is comparatively easy to trace jet-waves corresponding to various values of field-length and amplitude. It is done by calculating the path of a series of particles characterised by the moment t_0 at which they enter the field. The particles are appropriately chosen equidistant, distributed over half a wave-length, i. e. the values t_0 are distributed evenly over half a period. Corresponding to each value of t_0 a series of equidistant values are ascribed to t in the formulae (8)

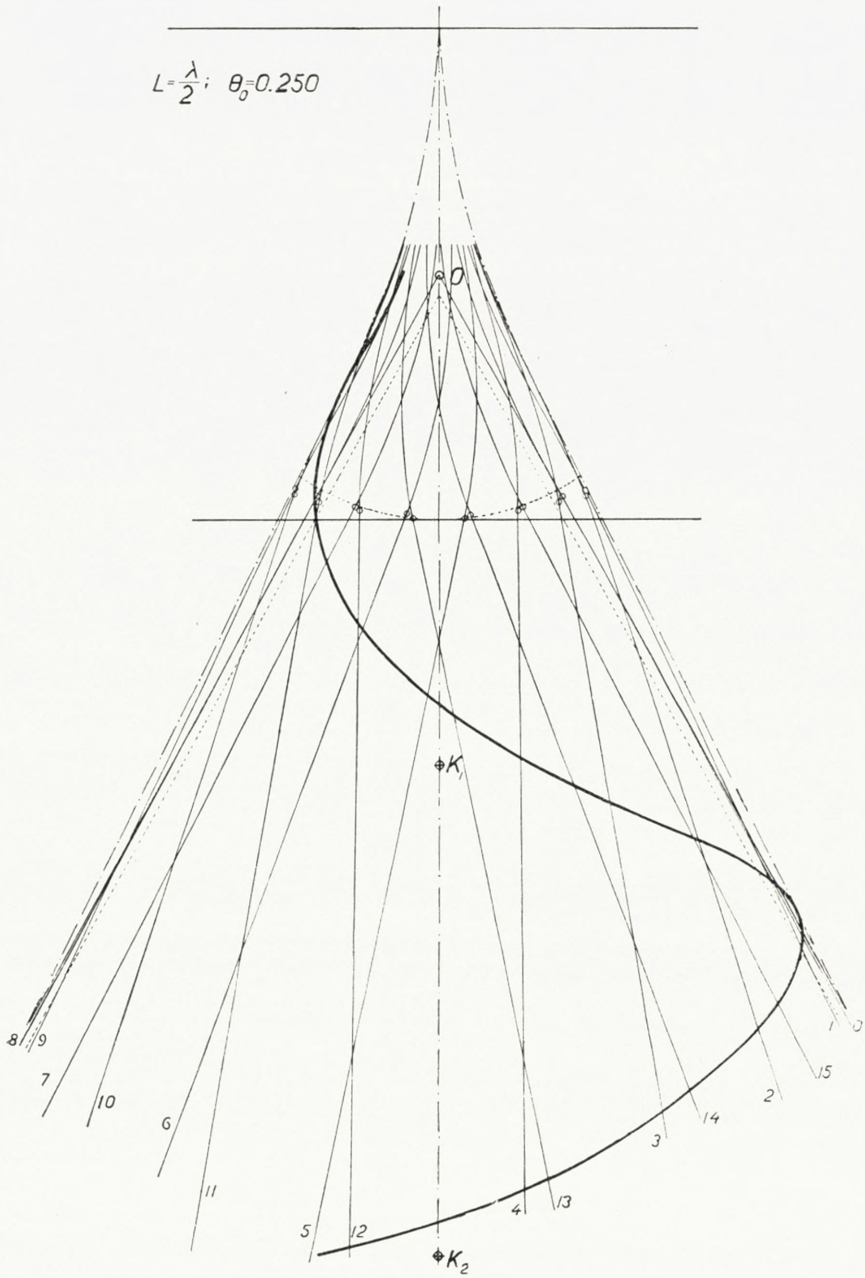


Fig. 18. Jet-wave Diagram $L = \frac{\lambda}{2}$, $\theta_0 = 0.250$.

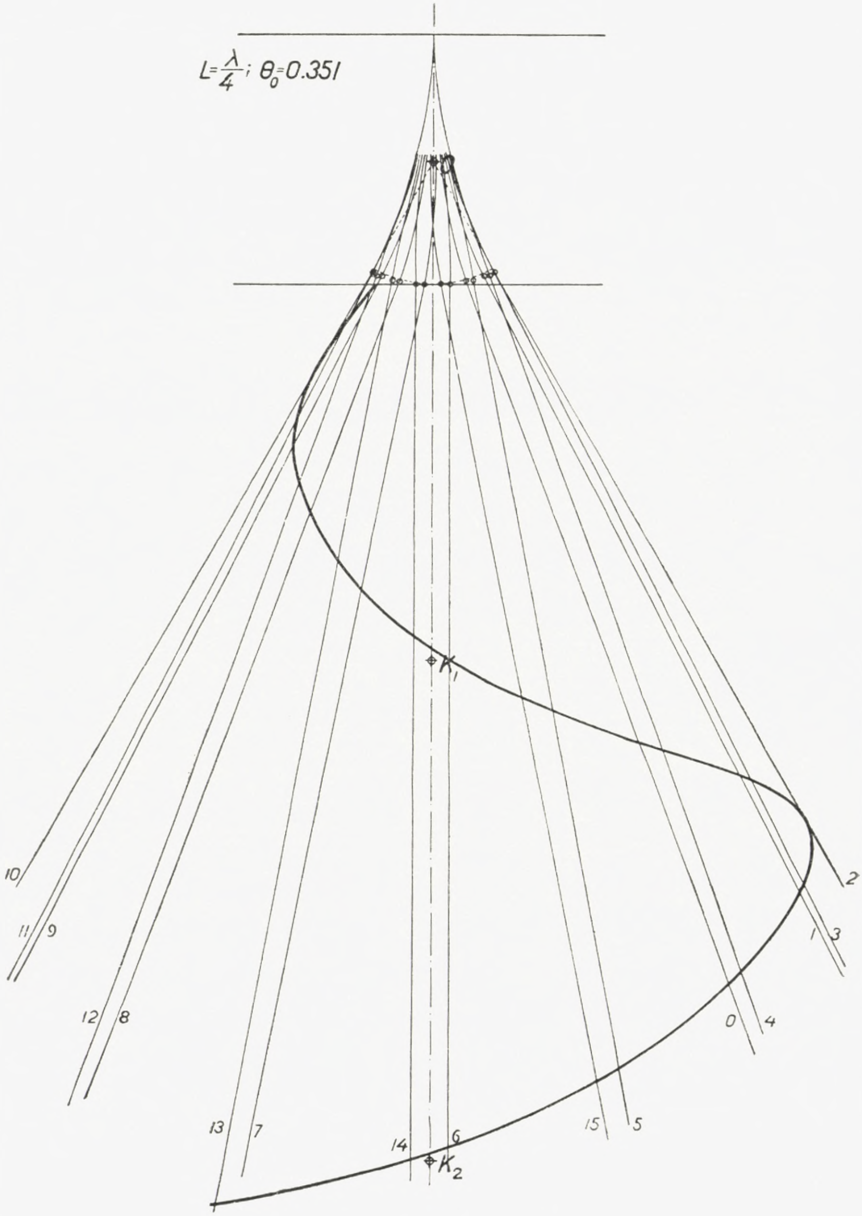


Fig. 19. Jet-wave Diagram $L = \frac{\lambda}{4}$, $\theta_0 = 0.351$.

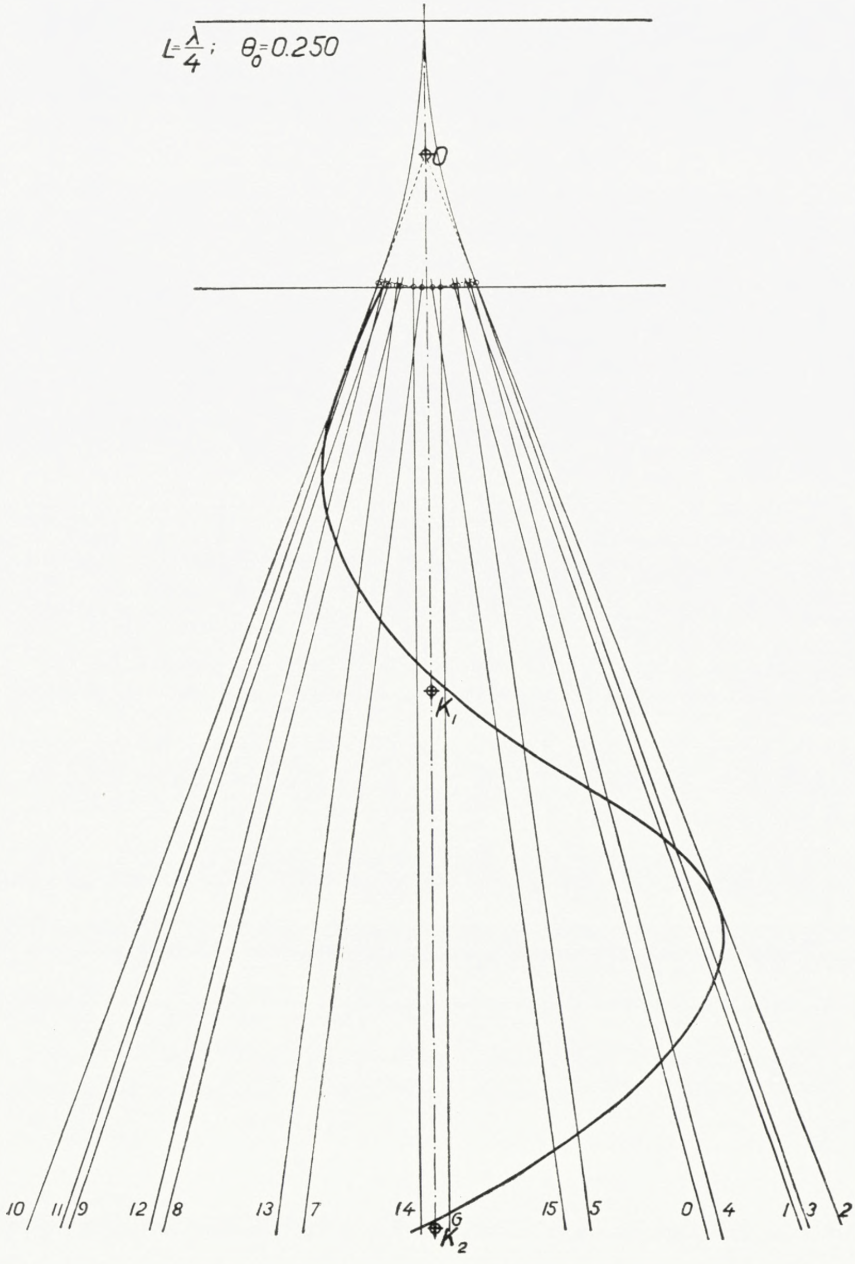


Fig. 20. Jet-wave Diagram $L = \frac{\lambda}{4}$, $\theta_0 = 0.250$.

and (11) of paragraph 3, and so a series of x- and y-values of the path is found. Furthermore the moment of arrival t_u of the particle at the lower boundary of the field is found by extra- or interpolation. If then t_u is introduced in (4) paragraph 3, the direction of the outside rectilinear path is determined. It being known that the particle moves on in the path with the velocity v of the original jet, it is also known at what point of the path the particle is found at any moment. If now the positions of a series of particles at a given moment are marked, the wave at the said moment may be drawn by tracing a curve through the said positions.

In the way here indicated the wave-pictures in fig. 18—20 have been produced. The direction of the paths outside the fields are stated in tab. I. Fig. 18 corresponds to a field of half a wave-length and to $\theta_0 = 0.250$. The paths numbered 0, 1, 2, ..., 15, correspond to particles entering the field at the moments 0, $1 \cdot \frac{T}{16}$, $2 \cdot \frac{T}{16}$, ..., $15 \cdot \frac{T}{16}$. (The current is supposed to be $i = I \sin \omega t$). On the paths the positions at which the particles are found half a period after their entrance into the field are marked by circles. It thus takes a little more than half a period for a particle to pass the field, and the more time the more sloping the path is. The wave proceeds within certain symmetrical boundary-curves, the amplitude curves, which of course are envelopes of the outermost paths. In fig. 18 the said curves are stippled. Their points have been determined graphically by measuring at a series of distances from the centre of the field the distances y from the axis of the wave to the various paths. The values of y have then been marked out in a rectangular system of coordinates in their dependency of the number of the path, and so the maximum-value of y has been found. In fig. 18 the amplitude-curves deviate markedly from the outermost paths. In fig. 19 cor-

Table I.

$\frac{t_0}{\left(\frac{T}{16}\right)}$	$\frac{t'}{\left(\frac{T}{16}\right)}$	θ	$tg\,\theta$	
0	8.44	0.496	0.542	
1	9.43	0.443	0.475	$L = \frac{\lambda}{2}$, $\theta_0 = 0.250$
2	10.52	0.314	0.324	
3	11.28	0.166	0.168	$tg\,\theta_m = 0.546$
4	12.14	0.013	0.013	Path $t_0 = -0.25\frac{T}{16}$
5	13.02	— 0.195	— 0.201	
6	14.10	— 0.360	— 0.376	$\theta_m = 0.500$
7	15.25	— 0.470	— 0.508	
0	4.06	0.360	0.376	
1	5.12	0.473	0.512	$L = \frac{\lambda}{4}$, $\theta_0 = 0.351$
2	6.20	0.515	0.566	
3	7.17	0.466	0.503	$tg\,\theta_m = 0.566$
4	8.13	0.350	0.366	Path $t_0 = 2\frac{T}{16}$
5	9.06	0.186	0.189	
6	10.00	0.000	0.000	$\theta_m = 0.515$
7	11.02	— 0.193	— 0.195	
0	4.04	0.253	0.259	
1	4.06	0.333	0.346	$L = \frac{\lambda}{4}$, $\theta_0 = 0.250$
2	4.10	0.360	0.377	
3	4.10	0.330	0.343	$tg\,\theta_m = 0.377$
4	4.06	0.250	0.255	Path $t_0 = 2\frac{T}{16}$
5	4.04	0.134	0.134	
6	4.01	— 0.001	— 0.001	$\theta_m = 0.360$
7	4.00	— 0.135	— 0.136	

responding to a field-length $\frac{\lambda}{4}$ the amplitude-curves coincide almost exactly with the outermost paths, for which t_0 is $2\frac{T}{16}$ and $10\frac{T}{16}$ respectively, so the amplitude curves are not drawn. The same is true in the case shown in fig. 20, where the outermost paths likewise correspond to t_0 equal to $2\frac{T}{16}$ and $10\frac{T}{16}$ respectively. In the table θ_m and $tg\,\theta_m$ indicate maximum-values of θ and $tg\,\theta$ respectively.

5. Geometric Construction of the Jet-Wave in the Case of a non-laminar Field.

The paths of the particles and so the wave may also be approximately determined purely geometrically as indicated in fig. 21. The field is divided into conveniently thin laminae or zones 1, 2, 3, ... Inside each of the said zones the path is assumed to be a circle with a radius determined by

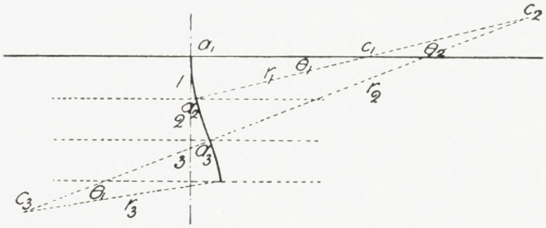


Fig. 21. Geometric Construction of Wave.

$$(1) \qquad \frac{1}{\varrho} = \frac{1}{10} \cdot \frac{iH}{m\,v^2},$$

i being the average value of the current during the passage of the zone. The centre of curvature C_1 for the path through the first zone is situated in the uppermost boundary of the field. The centre of curvature C_2 for the path through zone 2 is assumed to be on the line $C_1 a_2$, a_2 being the last point of the path inside zone 1 etc. In fig. 22 a—b an example of the construction is given, fig. 22 a showing 16 equidistant paths inside the field and fig. 22 b giving the corresponding paths outside the field and a complete wave-picture. In the construction the field was divided into 8 zones and it was assumed that the passage of each zone took $\frac{T}{16}$ sec. The current was supposed to be $i = \sin \omega t$ and the radius of curvature, measured in cm, was chosen 10 times the reciprocal value of $\sin \omega t$ at the moment at which the particle is in the middle of the zone in question. Furthermore λ was taken to be 16 cm. With these dimensions θ_0 is very nearly the same as in fig. 18, namely ab. 0.250. For to the factor 10 corresponds the value 1 of $\frac{HI}{m\,v^2}$ and

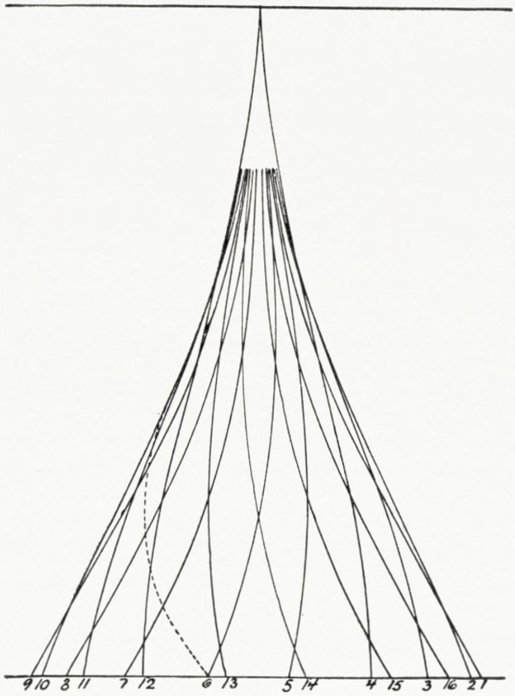


Fig. 22 a. Instance of Construction.

thus with $\lambda = 16$, the value of $\frac{1}{10}\,\frac{HI}{m\,v^2}\cdot\frac{\lambda}{2\pi} = \theta_0$ is $\frac{1}{10}\cdot\frac{\lambda}{2\pi} =$ 0.254. Very nearly the same paths and the same wave should therefore be expected in fig. 18 and in fig. 22 a—b. In a comparison it was found that path 1 in the original construction fig. 22 a cuts the lower boundary of the field 5.26 cm. from the axis of the wave, while the corresponding distance in fig. 18 was 5.16 cm., the difference being thus ab. 2 per cent. i. e. the same as between the values of θ_0.

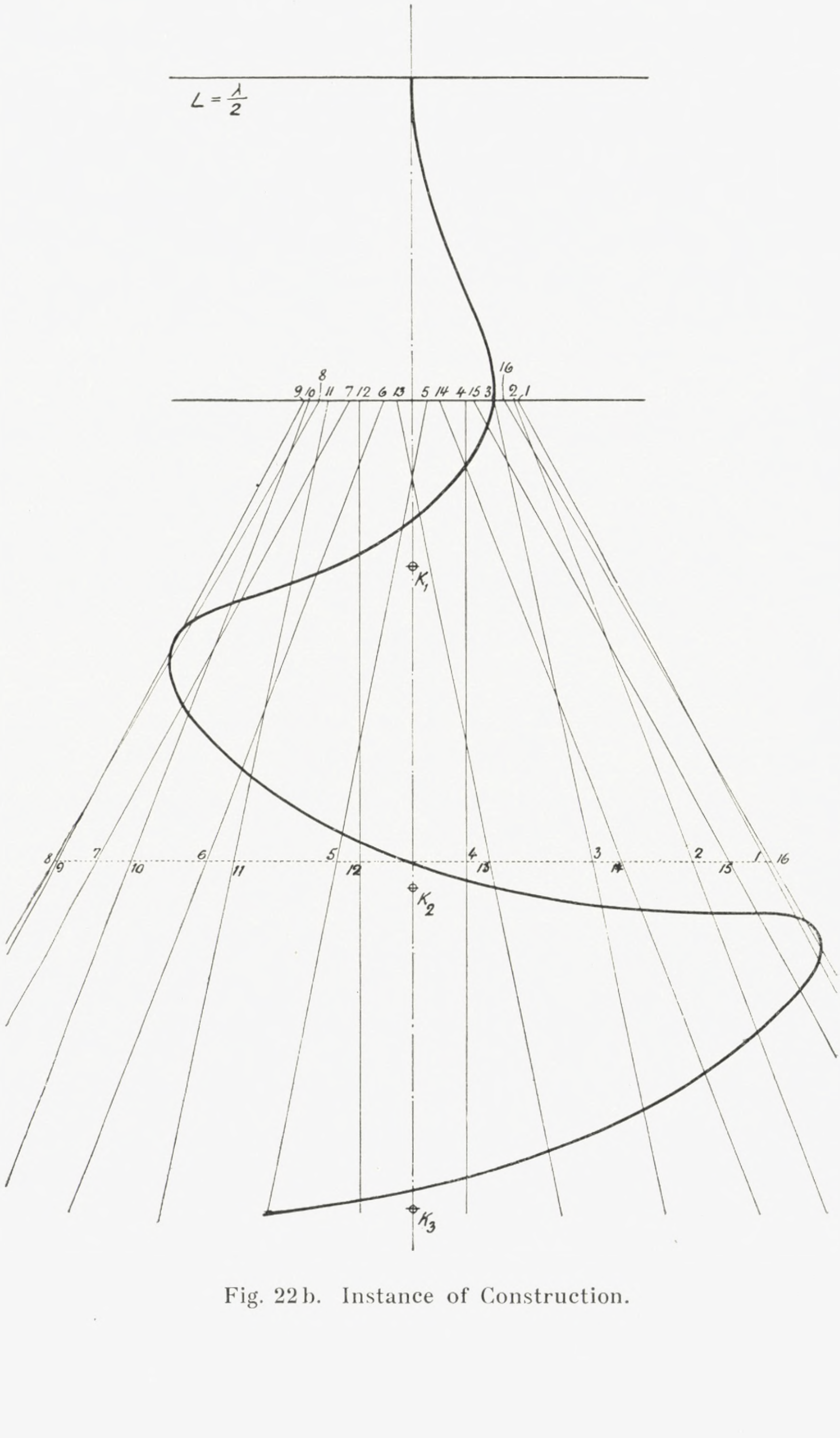


Fig. 22b. Instance of Construction.

The influence of the width of the zones in the geometric construction was examined. It was found that in a field of $\frac{\lambda}{2}$ the slope of path 1 outside the field varied as follows with the number of zones.

Table II.

Number of Zones with field of $\frac{\lambda}{2}$	$tg\,\theta$	θ
4	0.551	0.504
8	0.536	0.493
16	0.517	0.477
∞	0.500	0.463

The values of $tg\,\theta$ and θ corresponding to an infinite number of zones was found by means of an extrapolation of rather large uncertainty. From the preceding paragraph we conclude that the true value of θ should be ab. 0.452. It thus seems that the construction employing zones of $\frac{\lambda}{16}$ in the case considered, that is to say with a rather large amplitude, leads to a comparatively large error, say 8—10 per cent., in the determination of the slope of the paths and thereby also in the determination of the amplitude. Otherwise the constructive method has the advantage of affording a general means for the determination of the wave also in cases where the field is not homogeneous.

6. Approximate Theory of the Jet-Wave with a non-laminar Field.

If the non-laminar field is not too long in the direction of the jet, say not longer than $\frac{\lambda}{4}$, and if furthermore the amplitude is not too large, compare tab. IV, the equation of the wave may with fairly good exactness be determined as follows.

It is assumed that the wave will be nearly the same as would be produced if the current during the passage of each particle were constant and equal to the average value of the actual current during the said passage. Furthermore it is assumed that the passage of all particles takes the same time, namely $\frac{L}{v}$, thus the time for the passage of a particle of the original jet. For the particle which at the moment t_0 is at the middle-plane of the field the said average value of the current is

$$(1) \qquad I_g = \frac{1}{\gamma \frac{T}{2}} \cdot \int_{t_0 - \gamma \frac{T}{4}}^{t_0 + \gamma \frac{T}{4}} I \sin \omega t \cdot dt = I \frac{\sin \gamma \frac{\pi}{2}}{\gamma \frac{\pi}{2}} \cdot \sin \omega t_0$$

γ standing for $\frac{L}{\left(\frac{\lambda}{2}\right)}$.

The slope of the path due to I_g in interaction with the field is

$$(2) \qquad \sin \theta = \frac{1}{10} \cdot \frac{IH}{m v^2} \cdot L \cdot \frac{\sin \gamma \frac{\pi}{2}}{\gamma \frac{\pi}{2}} \cdot \sin \omega t_0 .$$

Prolonged backwards the path will generally intersect the axis of the original jet nearly at the centre of the field. If therefore the distance from the said point to an arbitrary point of the wave is indicated by r, it may be concluded from (2) when compared with (6) paragraph 1 that the equation of the jet-wave may be written

$$\text{(3)} \qquad \sin\theta = \frac{1}{10}\cdot\frac{IH}{m\,v^2}\cdot L\cdot\frac{\sin\gamma\frac{\pi}{2}}{\gamma\frac{\pi}{2}}\sin\omega\left(t-\frac{r}{v}\right)$$

$$= \sin\theta_m\cdot\sin\omega\left(t-\frac{r}{v}\right).$$

The angular amplitude θ_m of the wave is thus determined by

$$\text{(4)} \qquad \sin\theta_m = \frac{1}{10}\cdot\frac{IH}{m\,v^2}\cdot L\cdot\frac{\sin\gamma\frac{\pi}{2}}{\gamma\frac{\pi}{2}}.$$

or it is the same as with a fictive laminar field of length $L\cdot\frac{\sin\gamma\frac{\pi}{2}}{\gamma\frac{\pi}{2}}$. As in the case of waves with small amplitudes we find with waves of larger amplitudes that the wave with non-laminar field may be considered identical with waves produced by a laminar field as long as the factor $\frac{\sin\gamma\frac{\pi}{2}}{\gamma\frac{\pi}{2}}$ may be considered small compared with 1. Some values of the said factor are given in tab. III.

Table III.

γ	$\frac{\sin\gamma\frac{\pi}{2}}{\gamma\frac{\pi}{2}}$
0.1	0.994
0.2	0.984
0.3	0.962
0.4	0.936
0.5	0.902

7. Comparison between the general and the approximate Theory.

In Tab. IV below, the absolute amplitude y at various distances β from the centre of the field, and calculated from the approximate theory in the preceding paragraph, is compared with the corresponding amplitude measured in the wave pictures in fig. 18—20. The latter amplitude is stated under A, the former under B. The B-figures are determined from

$$y = \beta \cdot \frac{\lambda}{2} \cdot tg\, \theta_m, \tag{1}$$

where θ_m is found from (4) paragraph 6. Obviously the approximate theory agrees excellently with the general theory for field-lengths up to $\frac{\lambda}{4}$ and for amplitudes $\alpha = tg\, \theta_m$ up to 0.5 or even above.

Table IV.

	$\beta = \frac{x}{\left(\frac{\lambda}{2}\right)}$	A	B	
I	0.50	5.21	4.62	$L = \frac{\lambda}{2}$, $\lambda = 32$ cm.
	0.75	7.15	6.91	
	1.00	9.13	9.22	$2\,\theta_0 = \frac{1}{10} \cdot \frac{IH}{m\,v^2} \cdot \frac{\lambda}{\pi} = 0.500$
	1.50	13.18	13.84	
	2.00	17.44	18.47	$tg\, \theta_m = 0.546$ with $t_0 = -0.25 \cdot \frac{T}{16}$
II	0.50	4.50	4.58	$L = \frac{\lambda}{4}$, $\lambda = 32$ cm.
	0.75	6.74	6.85	
	1.00	9.02	9.12	$2\,\theta_0 = \frac{1}{10} \cdot \frac{IH}{m\,v^2} \cdot \frac{\lambda}{\pi} = 0.702$
	1.50	13.60	13.72	
	2.00	18.12	18.29	$tg\, \theta_m = 0.566$ with $t_0 = 2\,\frac{T}{16}$
III	0.50	3.03	3.02	$L = \frac{\lambda}{4}$, $\lambda = 32$ cm.
	0.75	4.53	4.54	
	1.00	6.04	6.04	$2\,\theta_0 = \frac{1}{10} \cdot \frac{IH}{m\,v^2} \cdot \frac{\lambda}{\pi} = 0.500$
	1.50	9.07	9.06	
	2.00	12.10	12.10	$tg\, \theta_m = 0.377$ with $t_0 = 2\,\frac{T}{16}$

We will now consider the question of the nodes. The wave represented by the approximate theory has its nodes at distances from the centre of the field given by $p \cdot \frac{\lambda}{2}$, where p stands for 1, 2, ... In figs. 18—20 these points are marked as K_1, K_2. With the field-length $\frac{\lambda}{2}$, fig. 18, K_1 and K_2 are displaced rather considerably with regard to the nodes of the actual wave, that is to say, the points of intersection with the axis, the wave being represented at the moment $t = 0$. And it may be noted that K_1 and K_2 are farther from the field than the actual nodes. With the shorter field $\frac{\lambda}{4}$ in fig. 19 the displacement of K_1 and K_2 is much smaller, and the same is true in the case shown in fig. 20, in which the field-length is also $\frac{\lambda}{4}$ while the amplitude is essentially less than in fig. 19. With field-lengths below $\frac{\lambda}{2}$ the nodes have very nearly the same positions as with a laminar field, and obviously the said positions practically do not depend on the amplitude.

It is also of some interest to compare the positions of the nodes determined from the general theory with the positions as found from the theory in chapter I for small

Table V.

Nodes	Length of Field	General Theory	Small Ampl. Theory
1	$\frac{\lambda}{2}$	0.878	0.894
2	$\frac{\lambda}{2}$	1.934	1.950
1	$\frac{\lambda}{4}$	0.980	0.981
2	$\frac{\lambda}{4}$	1.988	1.988

amplitudes. A comparison with a field-length of $\frac{\lambda}{2}$ is made in fig. 17. It may be supplemented by the figures in tab. V.

The figures given are the distances from the centre of the field measured in half wave-lengths. Those of column 3 are derived from fig. 18—20. Obviously the small amplitudes give very nearly the same positions of the nodes as the general theory, from which again may be concluded that the position of the nodes depends very little on the amplitude, a fact of great importance in the application of the waves in certain commutators.

8. The Jet-Wave with an inhomogeneous Field. The effective Length of the Field.

In all the cases considered above, the field was assumed to be homogeneous inside the space between the pole-pieces

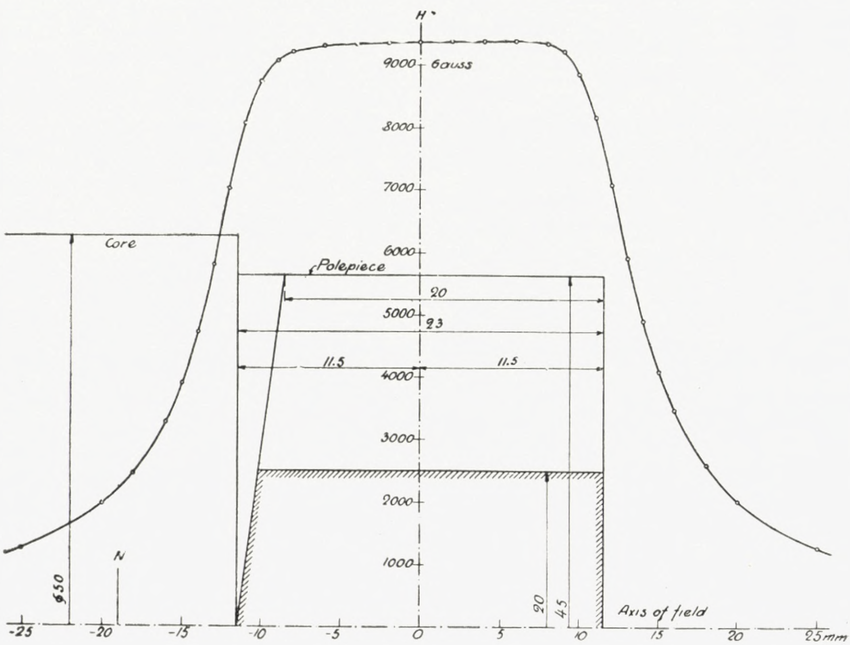


Fig. 23. Actual Field.

and of zero intensity outside the same. The actual magnetic fields are not homogeneous, the intensity varying along the axis of the jet in a way of which fig. 23 may convey an idea. The picture originates from a magnet the width of the field of which was 6.4 mm while the height of the pole-pieces in the direction of the jet was 23 mm and the maximum intensity of the field 9350 Gauss. The contour of half of a pole-piece is indicated by hatching. Obviously the field already commences to decrease inside the space between the pole-pieces. On the other hand a considerable stray-field is present outside the latter. The result hereof is that as a rule the field acts as a homogeneous field of a greater extension than the height L of the pole-pieces, even if an intensity equal to the maximum value of the actual field is ascribed to the fictive homogeneous field. The length L_e of the latter may be spoken of as the effective length of the actual field. According to what has been stated we may write

(1) $$L_e = L + \Delta L .$$

We now proceed to show how L_e or ΔL may be derived in cases where the field-curve of the actual field and the wave-length of the jet-wave are known.

Passing a zone of the extension dx in the direction of the original jet, fig. 24 a, the path of a particle suffers a change of direction $d\theta$ determined (compare paragraph 3) by

(2) $$d\theta = \frac{ds}{\varrho} = \frac{1}{10} \cdot \frac{iH}{m\, v^2} \cdot ds ,$$

i being the value of the current during the passage and H the intensity of the field within the zone in question. Furthermore

(3) $$dx = ds \cdot \cos\theta .$$

Hence

(4) $$d\theta = \frac{1}{10} \cdot \frac{iH}{m\,v^2} \cdot \frac{dx}{\cos\theta}$$

or

(5) $$d(\sin\theta) = \frac{1}{10} \cdot \frac{iH}{m\,v^2}\, dx.$$

We shall now, as in the development of the approximate theory in paragraph (6), assume that the amplitude

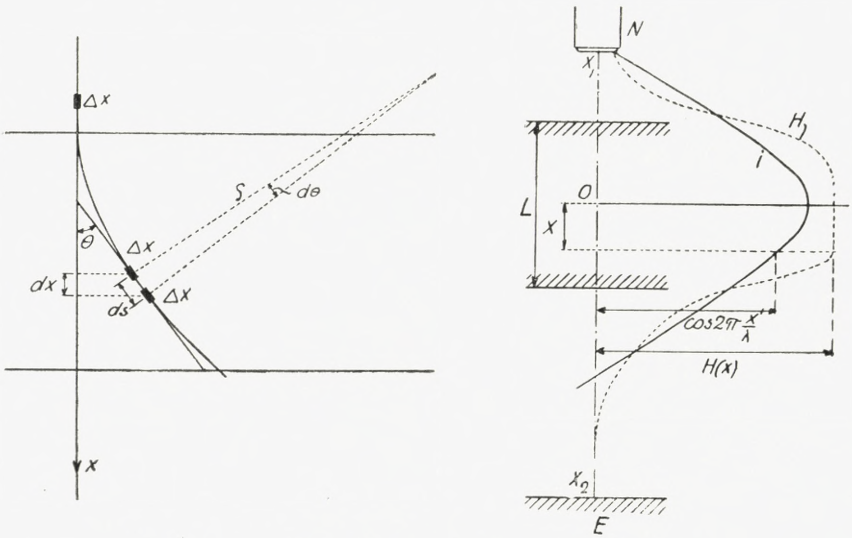


Fig. 24 a—b. Calculation of effective Field-Length.

is determined by the path of greatest slope and that the latter corresponds to the particle which is at the centre of the field at the moment of maximum current. Furthermore we shall assume that the time it takes for a particle to pass the zone dx is $\frac{dx}{v}$. A cosine-curve i, fig. 24 b, covering half a wave-length $\frac{\lambda}{2}$ is drawn with its top over the centre of the field O. Under the circumstances assumed it represents the variation of the current during the passage of the particle the path of which determines the amplitude θ_m. The latter amplitude is now calculated from

$$\sin\theta_m = \frac{1}{10}\cdot\frac{I}{m\,v^2}\int_{x_1}^{x_2} H(x)\cos\frac{2\pi}{\lambda}x\cdot dx, \tag{6}$$

where I stands for the maximum value of the current and where $H(x)$ and $\cos\frac{2\pi}{\lambda}x$ are read on the curves in fig. 24 b. The integration is to be taken from the abscissa x_1 of the nozzle to a point x_2 where the field intensity is practically zero. The amplitude given by (6) is now identified with the amplitude produced by the fictive homogeneous field of length $L_e = \gamma_e\cdot\frac{\lambda}{2}$ and with the maximum intensity H of the actual field. The latter amplitude is determined by

$$\begin{aligned}\sin\theta_m &= \frac{1}{10}\cdot\frac{IH}{m\,v^2}L_e\cdot\frac{\sin\gamma_e\frac{\pi}{2}}{\gamma_e\frac{\pi}{2}}\\ &= \frac{1}{10}\cdot\frac{IH}{m\,v^2}\cdot\frac{\lambda}{\pi}\cdot\sin\gamma_e\frac{\pi}{2}.\end{aligned} \tag{7}$$

Hence from (6) and (7) we get for the determination of γ_e

$$\sin\gamma_e\frac{\pi}{2} = \frac{1}{H\frac{\lambda}{\pi}}\int_{x_1}^{x_2} H(x)\cos\frac{2\pi}{\lambda}x\cdot dx. \tag{8}$$

With a view to illustration the effective length of the field was calculated for the case in fig. 23 and for a wave-length of 13.30 cm. The nozzle was assumed to be at the abscissa -1.9 cm and on the other side the integration was carried down to $+4.3$ cm where a prolongation of the curve, not given in the figure, showed the field-intensity to be negligible. By means of an integration in which dx was chosen equal to 0.2 cm. it was found that

$$\int_{-1.9}^{+4.3} H(x)\cos 2\pi\frac{x}{\lambda}\cdot dx = 25770$$

from which

$$\sin \gamma_e \frac{\pi}{2} = \frac{\pi}{13.30} \cdot \frac{25770}{9350} = 0.651,$$

$\gamma_e \cdot \frac{\pi}{2} = 0.709$, $\gamma_e = 0.451$ and $L_e = 0.451 \cdot 6.65 = 3.00$ cm, while L was 2.30 cm and thus $\Delta L = 0.70$ cm.

9. The effective Field-Length with stationary Deflection of the Jet.

It is of interest to determine the effective field-length in case of the jet carrying a constant current I so that a stationary deflection of the jet is produced. As seen from (5) in the preceding paragraph, the said deflection is given by

$$\sin \theta_d = \frac{1}{10} \cdot \frac{I}{m\, v^2} \int_{x_1}^{x_2} H(x)\, dx, \tag{1}$$

while with a homogeneous field of extension L_d it would be

$$\sin \theta_d = \frac{1}{10} \cdot \frac{IH}{m\, v^2} \cdot L_d. \tag{2}$$

Equalizing (1) and (2) and taking H to indicate the maximum-value of the actual field we find for the effective field-length L_d with a direct current through the jet

$$L_d = \frac{1}{H} \int_{x_1}^{x_2} H(x)\, dx. \tag{3}$$

If $\sin \theta_d$ taken from (2) is introduced in (7) of the preceding paragraph we find

$$\sin \theta_m = \sin \theta_d \cdot \frac{L_e}{L_d} \cdot \frac{\sin \gamma_e \frac{\pi}{2}}{\gamma_e \frac{\pi}{2}}, \tag{4}$$

from which it is seen that the amplitude with an alternating current is no longer identical with the stationary deflection produced by a direct current equal to the maximum-value of the alternating current as in the case with a short homogeneous field. In calculating $\sin\theta_m$ we have to multiply $\sin\theta_d$ with two factors, of which the one $\dfrac{\sin\gamma_e\frac{\pi}{2}}{\gamma_e\frac{\pi}{2}}$ originates from the field not being a laminar field while the other $\dfrac{L_e}{L_d}$ is due to the field not being homogeneous.

In the case of fig. 23 we find $L_d = \dfrac{29490}{9350} = 3.15$ cm, thus appreciably different from the effective field-length with a wave-length of 13.3 cm, the factor $\dfrac{L_e}{L_d}$ being $\dfrac{3.00}{3.15} = 0.952$. With the same field and wave-length $\dfrac{\sin\gamma_e\frac{\pi}{2}}{\gamma_e\frac{\pi}{2}} = \dfrac{0.651}{0.709} = 0.919$ so that $\sin\theta_m = 0.952\cdot 0.919\cdot\sin\theta_d = 0.875\sin\theta_d$.

10. Damping of the Wave.

The theories stated above have all been based on the assumption that the several particles of the jet are independent of each other in their motion. Now, actually two neighbouring elements do influence each other and the influence may probably be conceived in the way illustrated in fig. 25. Here the original jet is considered as made up of disks. When a wave is formed, these disks are displaced with regard to each other. Thus the viscosity η of the fluid comes into action and will cause the amplitude to be somewhat smaller than predicted by the elementary theories above.

The wave considered in fig. 25 is of the rectangular type indicated in chapter I, and we shall here confine ourselves to that type of wave. We may consider three adjacent

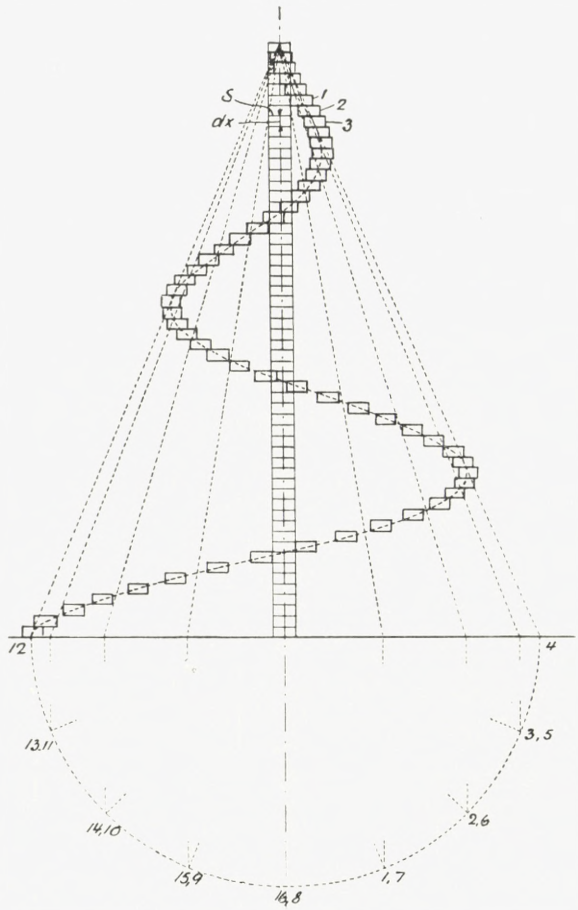


Fig. 25. Damping of Jet-Wave.

elements 1, 2, 3 of which 2 passes the laminar field of extension dl at the moment t_0 at which the alternating current producing the wave is

$$i = I \sin \omega t_0 . \tag{1}$$

In the field the element in question obtains a velocity perpendicular to the original direction of the jet given by

$$v_y = \frac{1}{10} \cdot \frac{HI}{\varrho S v} \cdot dl \cdot \sin \omega t_0 , \tag{2}$$

where ϱ stands for the density of the liquid while S is the area of the cross-section of the jet. Outside the field the element 2 is acted on by tangential forces in the surfaces of separation between 2 and the adjacent elements 1 and 3. The force originating from 1 may be written

$$K = -\eta S \cdot \frac{dv_y}{dx} \tag{3}$$

and from the element 3

$$K + \varDelta K = \eta S \left[\frac{dv_y}{dx} + \frac{d}{dx} \left(\frac{dv_y}{dx} \right) dx \right]. \tag{4}$$

The resultant force is thus

$$\varDelta K = \eta S \cdot \frac{d^2 v_y}{dx^2} \cdot dx. \tag{5}$$

We now take t_0 as the independent variable instead of x noting that

$$\frac{dv_y}{dx} = \frac{dv_y}{dt_0} \cdot \frac{dt_0}{dx} = -\frac{1}{v} \cdot \frac{dv_y}{dt_0} \tag{6}$$

and thus

$$\frac{d^2 v_y}{dx^2} = \frac{1}{v^2} \cdot \frac{d^2 v_y}{dt_0^2}. \tag{7}$$

From (2) it follows that

$$\frac{d^2 v_y}{dt_0^2} = -\frac{1}{10} \cdot \frac{HI}{\varrho S v} \cdot dl \cdot \omega^2 \cdot \sin \omega t_0. \tag{8}$$

Hence

$$\varDelta K = -\frac{1}{10} \eta \cdot \frac{HI}{\varrho v^3} \cdot dl \cdot \omega^2 \cdot \sin \omega t_0 \cdot dx. \tag{9}$$

The motion produced by $\varDelta K$ is determined by

$$m \frac{d^2 y}{dt^2} = -\frac{1}{10} \eta \cdot \frac{HI}{\varrho v^3} \cdot dl \cdot \omega^2 \cdot \sin \omega t_0 \tag{10}$$

from which

$$(11)\qquad y = -\frac{1}{10}\cdot\frac{\eta}{m}\cdot\frac{HI}{\varrho v^3}\cdot dl\cdot\omega^2\cdot\sin\omega t_0\cdot\frac{t^2}{2} + c_1\cdot t + c_2,$$

c_1 and c_2 being arbitrary constants. Their values are derived from the conditions

$$y = 0 \text{ and } \frac{dy}{dt} = v_y \text{ at the moment } t = t_0$$

from which

$$(12)\qquad \begin{aligned} c_1 = {} & \frac{1}{10}\cdot\frac{\eta}{m}\cdot\frac{HI}{\varrho v^3}\cdot dl\cdot\omega^2\cdot\sin\omega t_0 + \\ & \frac{1}{10}\cdot\frac{HI}{\varrho S v}\cdot dl\cdot\sin\omega t_0 . \end{aligned}$$

$$(13)\qquad \begin{aligned} c_2 = {} & \frac{1}{10}\,\frac{\eta}{m}\cdot\frac{HI}{\varrho v^3}\cdot dl\cdot\omega^2\sin\omega t_0\cdot\frac{t_0^2}{2} \\ & -\frac{1}{10}\cdot\frac{\eta}{m}\cdot\frac{HI}{\varrho v^3}\cdot dl\cdot\omega^2\cdot\sin\omega t_0\cdot t_0^2 - \frac{1}{10}\cdot\frac{HI}{\varrho S v}\cdot dl\cdot\sin\omega t_0\cdot t_0 , \end{aligned}$$

which introduced in (11) give

$$(14)\qquad y = \frac{1}{10}\,\frac{HI}{m v^2}\cdot dl\cdot x\cdot\sin\omega\left(t - \frac{x}{v}\right)\left[1 - \frac{\eta\omega^2}{2\varrho v^3}\,x\right]$$

it being noted that $x = v(t - t_0)$,

Now $\frac{1}{10}\cdot\frac{HI}{m v^2}\cdot dl\cdot x$ is the amplitude y_m predicted by the elementary theory (in the case of small amplitudes). The actual amplitude may be written

$$(15)\qquad y = f\cdot\frac{1}{10}\cdot\frac{HI}{m v^2}\cdot dl\cdot x,$$

where f is termed the inverse damping factor. It is less than 1 and by (14) may be written

$$(16)\qquad f = 1 - \tfrac{1}{2}\,\frac{\eta}{\varrho}\cdot\frac{\omega^2}{v^3}\cdot x = 1 - \tfrac{1}{2}\,\nu\cdot\frac{\omega^2}{v^3}\cdot x$$

or

$$(17) \qquad f = 1 - 2\pi^2 \nu \cdot \frac{x}{v\lambda^2} = 1 - 2\pi^2 \nu \cdot T \cdot \frac{x}{\lambda} \cdot \frac{1}{\lambda^2},$$

$\nu = \frac{\eta}{\varrho}$ being the dynamical viscosity of the liquid.

Obviously the theory set forth above is based on the supposition of small damping in that $\frac{d^2 v_y}{dx^2}$ in (5) is derived from the expression (2) for the lateral velocity obtained in the field. It is thus assumed that the latter velocity is only altered by a very small amount by the damping forces. With the mercury jet-waves employed in jet-wave commutators the said assumption is as a rule justified and (17) should accordingly in the main represent the relation between the damping and the various quantities on which it may depend. Considering a mercury-wave, we have $\nu = 0.00116$ (at 18 Centigrades). Let $\frac{x}{\lambda} = 1$, $\lambda = 6$ cm and $v = 600$ cm per sec. then $2\pi^2 \nu \cdot \frac{x}{v\lambda^2} = 0.635 \cdot 10^{-5}$, i. e. very small compared to 1. Practically no damping should thus be expected. As a matter of fact the damping in a case like that considered is so small that it is difficult to measure it, and therefore also to test the theory in order to see whether the conception of fig. 25 holds good, or whether one is justified in using for η (or ν) the value corresponding to a laminar flow of the liquid. Obviously the theory now indicated does not take all forces into account. Thus also the surface-tension will undoubtedly give rise to damping. We shall not, however, go further into the problem of the said damping, it being proposed to subject the whole question to a special experimental investigation.

Provided, however, that the expression (17) holds good in the main, information of considerable interest may be derived from it. In the first place it is seen that the damping is independent of the diameter of the jet, a fact which is easily understood, for according to (5) the damping force acting on an element of the wave is proportional to the area S of the cross-section, but the mass of the element is proportional to the same quantity, and so the motion becomes independent of S. Furthermore it is seen that the damping observed at a distance which, measured in wave-lengths, has a definite value $\left(\frac{x}{\lambda} \text{ constant}\right)$, is inversely as λ^2. The damping thus probably increases very markedly with decreasing λ or, with constant velocity v, with increasing frequency. Finally the expressions (16) and (17) predict a damping for water which is ab. 10 times greater than for mercury, ν being ab. 10 times less for the latter liquid than for the former.

CHAPTER III

Particular Properties of the Jet-Wave.

1. Slope and Cross-Section. Rectangular Jet-Wave Type.

In the present chapter we shall consider a series of special properties of the rectangular and the circular jet-waves, discussed above.

The rectangular type may be represented by

$$(1)\qquad y = \alpha x \sin \omega \left(t - \frac{x}{v}\right) = \alpha x \sin(\omega t - \mu x),$$

α indicating tangens to the angle θ_m, fig. 26.

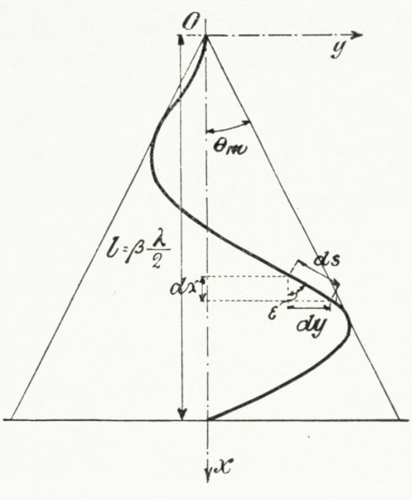


Fig. 26. Jet-Wave, rectangular Type.

The slope of an element ds of the wave against the x-axis is determined by

$$(2)\qquad tg\,\varepsilon = \frac{dy}{dx}$$

or from (1) by

$$(3)\qquad tg\,\varepsilon = \alpha \sin(\omega t - \mu x) - \mu \alpha x \cos(\omega t - \mu x).$$

The slope in a certain point x, y is found from (3) by first determining by means of (1) the moment t at which the wave passes the said point. The slope in the x-axis is of special interest. For points of the latter $y = 0$ i. e.

(4) $\sin(\omega t - \mu x) = 0$ and (5) $\cos(\omega t - \mu x) = \pm 1.$

Hence

$$(6) \qquad tg\,\varepsilon = \mp \mu \alpha x = \mp \pi \alpha \beta,$$

β standing for $\dfrac{x}{\left(\dfrac{\lambda}{2}\right)}$.

The element ds of the wave must contain the same amount of liquid as the element dx of the original jet, dx being the projection of ds on the x-axis. If therefore S_0 indicates the area of the cross-section of the original jet and S the corresponding quantity of the element ds, then

$$(7) \qquad S \cdot ds = S_0 \cdot dx$$

from which

$$(8) \qquad S = \frac{S_0}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}.$$

Hence the cross-section of the wave in the x-axis is

$$(9) \qquad S = \frac{S_0}{\sqrt{1 + \pi^2 \alpha^2 \beta^2}}$$

and the diameter

$$(10) \qquad d = \frac{d_0}{\sqrt[4]{1 + \pi^2 \alpha^2 \beta^2}},$$

d_0 being the diameter of the original jet.

For $\alpha = 0.5$, $\beta = 1.8$, (6) gives $tg\,\varepsilon = 2.83$, $\varepsilon = 70°30'$, while, from (9) $S = \dfrac{S_0}{2.99} = 0.333 \cdot S_0$ and $d = 0.577\, d_0$.

2. Slope and Cross-section. Circular Wave-type.

A wave of the circular type may generally be represented by

$$(1) \qquad \sin\theta = \sin\theta_m \sin\omega\left(t - \frac{r}{v}\right) = \sin\theta_m \sin(\omega t - \mu r).$$

From fig. 27 we see that the slope of an element ds towards the radius-vector r is determined by

(2) $$tg\,\varepsilon = \frac{r\,d\theta}{dr}.$$

Hence from (1)

(3) $$tg\,\varepsilon = -\frac{\sin\theta_m}{\cos\theta}\cdot\frac{r\,\omega}{v}\cos(\omega t - \mu r).$$

Again the slope in a given point r, θ is found by eliminating t from (1) and (3). For points in the axis of the original jet, $\theta = 0$,

(4) $\sin(\omega t - \mu r) = 0$

and

(5) $\cos(\omega t - \mu r) = \pm 1$

so that

(6) $$tg\,\varepsilon = \mp \sin\theta_m \cdot \frac{r\,\omega}{v} = \mp \pi\beta \sin\theta_m = \mp \pi\beta\alpha',$$

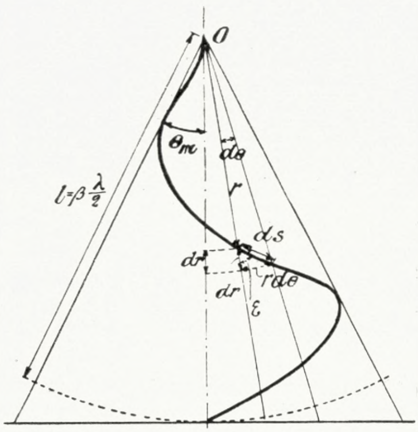


Fig. 27. Jet-Wave, circular Type.

if $\beta = \dfrac{r}{\left(\dfrac{\lambda}{2}\right)}$ and $\alpha' = \sin\theta_m$. The difference from the rectangular wave is thus that $\alpha = tg\,\theta_m$ is replaced by $\alpha' = \sin\theta_m$.

With the circular jet-wave the element ds of the wave contains the same volume of liquid as the element dr of the original jet, dr being now the circular projection of ds. Therefore

(7) $$S\,ds = S_0\,dr$$

or

(8) $$S = \frac{S_0}{\sqrt{1+\left(\dfrac{r\,d\theta}{dr}\right)^2}} = \frac{S_0}{\sqrt{1+tg^2\,\varepsilon}}$$

thus for points of the axis

(9) $$S = \frac{S_0}{\sqrt{1+\pi^2\alpha'^2\beta^2}}$$

and

(10) $$d = \frac{d_0}{\sqrt[4]{1+\pi^2\alpha'^2\beta^2}}.$$

For $\beta = 1.8$, $\alpha = tg\,\theta_m = 0.5$ i. e. $\alpha' = \sin\theta_m = 0.447$ we find from (6) $tg\,\varepsilon = 2.53$, $\varepsilon = 68°27'$ and from (8) and (10) $S = 0.367\,S_0$ and $d = 0.606\,d_0$. Under the same conditions the rectangular wave is thus a little more horizontal (vertical jet assumed) than the circular wave.

3. Electrical Resistance of the Wave. Rectangular Type.

Of very great importance in certain applications (jet-wave commutators) is the question of the ratio of the resistance of the wave and the resistance of the corresponding piece of the original jet. We start with the rectangular wave

(1) $$y = \alpha x \sin\omega\left(t - \frac{x}{v}\right) = \alpha x \sin(\omega t - \mu x).$$

The resistance of the wave from its starting-point $x = 0$ to the plane $x = l$, fig. 26, is determined by

(2) $$R = \int_0^l k\cdot\frac{ds}{\frac{\pi}{4}d^2} = \frac{k}{\frac{\pi}{4}d_0^2}\int_0^l \left(\frac{ds}{dx}\right)^2 dx$$

as

(3) $$d_0^2\,dx = d^2\cdot ds.$$

In (2) k indicates the specific resistance of the liquid. The resistance in the length l of the original jet is

(4) $$R_0 = \frac{k}{\frac{\pi}{4}d_0^2}\cdot l.$$

The ratio between the resistances of the undulating and non-undulating jet is thus

(5) $$F = \frac{R}{R_0} = \frac{1}{l}\int_0^l \left(\frac{ds}{dx}\right)^2 dx = 1 + \frac{1}{l}\int_0^l \left(\frac{dy}{dx}\right)^2 dx$$

or

(6) $$F - 1 = \frac{1}{l}\int_0^l \left(\frac{dy}{dx}\right)^2 dx.$$

From (1)

(7) $$\frac{dy}{dx} = \alpha \sin(\omega t - \mu x) - \mu\alpha x \cos(\omega t - \mu x),$$

hence

(8) $$\begin{aligned}\left(\frac{dy}{dx}\right)^2 &= \alpha^2 \sin^2(\omega t - \mu x) + \mu^2 \alpha^2 x^2 \cos^2(\omega t - \mu x) \\ &\quad - 2\mu\alpha^2 x \sin(\omega t - \mu x)\cos(\omega t - \mu x) \\ &= \alpha^2 \cdot \frac{1 - \cos 2(\omega t - \mu x)}{2} - \mu\alpha^2 x \sin 2(\omega t - \mu x) \\ &\quad + \mu^2 \alpha^2 x^2 \frac{1 + \cos 2(\omega t - \mu x)}{2}.\end{aligned}$$

The average value taken over the time of $F-1$ is found by carrying out the integration indicated in (6) for those members of (8) only which do not contain trigonometric functions as the latter members are bound to disappear, being periodic functions of time with a frequency twice that of the jet-wave so that the average taken over a period of the latter is zero. Hence the average value of $F-1$ is simply

(9) $$\overline{F-1} = \frac{\alpha^2 \mu^2 l^2}{6} + \frac{\alpha^2}{2}$$

or with $l = \beta \frac{\lambda}{2}$

(10) $$\overline{F-1} = \frac{\alpha^2 \beta^2 \pi^2}{6} + \frac{\alpha^2}{2}$$

it being noted that $\mu = \frac{2\pi}{\lambda}$.

The variations in the resistance of the jet-wave are found by carrying out the integration (6) for the periodic members of (8). The result may be written:

$$\begin{aligned}
&\Delta(F-1) \\
&= -\frac{\alpha^2}{4}\left[\frac{\sin\mu l}{\mu l}\cos 2(\omega t-\mu l)\right. \\
&\quad\left.+\mu l\sin 2(\omega t-\mu l)+\cos 2(\omega t-\mu l)\right] \\
(11) &= -\frac{\alpha^2}{4}\left[\left(\frac{\sin\pi\beta}{\pi\beta}\cos\pi\beta-\pi\beta\sin 2\pi\beta+\cos 2\pi\beta\right)\cos 2\omega t\right. \\
&\quad\left.+\left(\frac{\sin\pi\beta}{\pi\beta}\sin\pi\beta+\pi\beta\cos 2\pi\beta+\sin 2\pi\beta\right)\sin 2\omega t\right] \\
&= -\frac{\alpha^2}{4}(A\sin 2\omega t+B\cos 2\omega t)=-\frac{\alpha^2}{4}\sqrt{A^2+B^2}\sin(2\omega t+\varphi)
\end{aligned}$$

where $tg\varphi = \frac{B}{A}$. The amplitude of the periodic variations of double frequency is

$$\begin{aligned}
&\frac{\alpha^2}{4}\sqrt{A^2+B^2} \\
(12)\quad &= \frac{\alpha^2}{4}\sqrt{\left(\frac{\sin\pi\beta}{\pi\beta}\right)^2+(\pi\beta)^2+1-2\sin^2\pi\beta+2\frac{\sin\pi\beta}{\pi\beta}\cos\pi\beta} \\
&= \frac{\alpha^2}{4}\sqrt{\left(\frac{\sin\pi\beta}{\pi\beta}\right)^2+\cos 2\pi\beta+2\frac{\sin 2\pi\beta}{2\pi\beta}+(\pi\beta)^2}\,.
\end{aligned}$$

With increasing extension β of the wave the latter expression tends to $\frac{\alpha^2}{4}\cdot\pi\beta$ or to $\frac{\alpha^2\beta^2\pi^2}{4\pi\beta}$. Hence for large values of β $F-1$ may be expressed by

$$(13)\qquad F-1 = \frac{\alpha^2\beta^2\pi^2}{6}\left[1-\frac{3}{2}\cdot\frac{1}{\pi\beta}\sin(2\omega t+\varphi)\right]+\frac{\alpha^2}{2}$$

from which it is again seen that $F-1$ with increasing β tends to a value independent of time, namely the average value $\overline{F-1} = \frac{\alpha^2\beta^2\pi^2}{6}+\frac{\alpha^2}{2}$. In tab. I the function

$$F = 1 + \frac{\alpha^2 \beta^2 \pi^2}{6} + \frac{\alpha^2}{2} - \frac{\alpha^2}{4} \sqrt{A^2 + B^2} \sin (2 \omega t + \varphi)$$

is illustrated through numerical values of the constant and variable part, and of their ratio. The latter ratio measures the relative variations of the resistance of the jet-wave considered. It is seen that with $\alpha = 0.5$ and $\beta = 2$ the resistance of the wave varies with an amplitude of ab. 14 per cent. of the average resistance. Obviously, in the case considered, the relative amplitude of the resistance has a maximum for some value between $\beta = 2$ and $\beta = 10$.

Table I.

β	1. $\frac{\alpha^2}{4}\sqrt{A^2+B^2}$ $\alpha = 0.5$	2. $1 + \frac{\alpha^2\beta^2\pi^2}{6} + \frac{\alpha^2}{2}$	1./2.
0.5	0.0856	1.228	0.070
1.0	0.206	1.536	0.134
1.5	0.288	2.052	0.140
2.0	0.398	2.775	0.144
10.0	3.93	42.195	0.093

4. Resistance of Wave with constant Amplitude.

For the sake of completeness a formula for the resistance of a simple sine-shaped wave of constant amplitude may be derived. The wave may be represented by

(1) $$y = y_0 \sin (\omega t - \mu x)$$

from which

(2) $$\left(\frac{ds}{dx}\right)^2 = 1 + y_0^2 \mu^2 \cos^2 (\omega t - \mu x).$$

Hence from (5) in the preceding paragraph

$$(3)\qquad \begin{aligned} F-1 &= \frac{y_0^2\mu^2}{l}\int_0^l \cos^2(\omega t-\mu x)\,dx \\ &= \frac{y_0^2\mu^2}{2}-\frac{y_0^2\mu^2}{4\mu l}\left[\sin 2(\omega t-\mu l)-\sin 2\omega t\right]. \end{aligned}$$

The average value of $F-1$ over the time is thus

$$(4)\qquad \overline{F-1} = \frac{y_0^2\mu^2}{2} = \frac{\gamma^2\pi^2}{2},$$

where $\gamma = \frac{y_0}{\left(\frac{\lambda}{2}\right)}$.

The expression (3) may be written

$$(3')\qquad F-1 = \frac{y_0^2\mu^2}{2}\left[1+\frac{\sin\mu l}{\mu l}\cos(2\omega t-\mu l)\right].$$

The second member in the brackets is maximum or minimum at the moments determined by

$$(5)\qquad t = \frac{p+\beta}{4}\cdot T = \beta\frac{T}{4}+p\frac{T}{4} \text{ where } p = 0, 1, 2, \ldots$$

and where $\beta = \frac{l}{\left(\frac{\lambda}{2}\right)}$. The member is zero at the moments

$$(6)\qquad t = \frac{\frac{p}{2}+\beta}{4}\,T = \beta\frac{T}{4}+p\frac{T}{8} \text{ where } p = 1, 2, 3, \ldots$$

For special values of l, resp. β, $F-1$ is independent of time, thus for the values given by $\sin\mu l = 0$ (except $\mu l = 0$) i. e. when

$$(7)\qquad \mu l = p\cdot\pi,\ (p = 1, 2, 3, \ldots)$$

thus for

$$(7')\qquad \beta = p,\ (p = 1, 2, 3, \ldots)$$

The greatest variations in $F-1$ are obtained when l is determined by

(8) $$tg\,\mu l = tg\,\pi\beta = \pi\beta$$

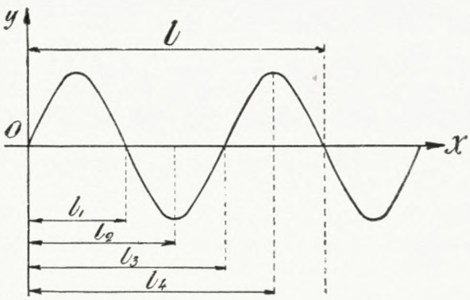
that is to say, approximately when

(9) $$\beta\pi = p\cdot\frac{\pi}{2},\ (p = 3,\ 5,\ 7)$$

thus for

(10) $$\beta = \frac{p}{2}.$$

Fig. 28 illustrates some of the relations indicated.



l_1, l_3 no variations in resistance.
l_2, l_4 maximum of variation.

Fig. 28. Jet-Wave, constant Amplitude.

5. Resistance of a Jet-Wave of circular Type.

We proceed to consider the resistance of a wave of the type in fig. 27. In this wave the element ds contains, as indicated, the same amount of liquid as the circular projection dr of the original jet. Hence the ratio of the resistance of the wave out to a circle with radius l, and the resistance of the length l of the original jet is now, compare paragraph 3,

(1) $$F = \frac{R}{R_0} = \frac{1}{l}\int_0^l \left(\frac{ds}{dr}\right)^2 dr$$

or as

$$(2) \qquad \left(\frac{ds}{dr}\right)^2 = 1 + \left(\frac{r\,d\theta}{dr}\right)^2$$

$$(3) \qquad F - 1 = \frac{1}{l}\int_0^l r^2 \left(\frac{d\theta}{dr}\right)^2 \cdot dr.$$

The wave may be represented by

$$(4) \qquad \sin\theta = \alpha' \sin(\omega t - \mu r).$$

From (4)

$$(5) \qquad r^2\left(\frac{d\theta}{dr}\right)^2 = \frac{\mu^2\alpha'^2 r^2 \cos^2(\omega t - \mu r)}{1 - \alpha'^2 \sin^2(\omega t - \mu r)}.$$

Thus

$$(6) \quad F - 1 = \frac{1}{l}\int_0^l \mu^2 \alpha'^2 r^2 \cos^2(\omega t - \mu r)\left[1 - \alpha'^2 \sin^2(\omega t - \mu r)\right]^{-1} dr.$$

By developing $\left[1 - \alpha'^2 \sin^2(\omega t - \mu r)\right]^{-1}$ in series (6) may be written

$$(7) \quad \begin{aligned} F - 1 = \frac{\mu^2\alpha'^2}{l}\int_0^l & r^2 \Big[1 + \alpha'^2 \sin^2(\omega t - \mu r) + \alpha'^4 \sin^4(\omega t - \mu r) \\ & + \alpha'^6 \sin^6(\omega t - \mu r) + \alpha'^8 \sin^8(\omega t - \mu r) - \sin^2(\omega t - \mu r) \\ & - \alpha'^2 \sin^4(\omega t - \mu r) - \alpha'^4 \sin^6(\omega t - \mu r) - \alpha'^6 \sin^8(\omega t - \mu r)\Big] dr. \end{aligned}$$

After this the powers of the trigonometric functions are expressed by trigonometric functions of multiples of $(\omega t - \mu r)$ by means of the following formulae.

$$\sin^2 z = \frac{1}{2}(1 - \cos 2z)$$

$$\sin^4 z = \frac{1}{8}(\cos 4z - 4\cos 2z + 3)$$

$$\sin^6 z = -\frac{1}{32}(\cos 6z - 6\cos 4z + 15\cos 2z - 10)$$

$$\sin^8 z = \frac{1}{128}(\cos 8z - 8\cos 6z + 28\cos 4z - 56\cos 2z + 35).$$

If, however, as is generally the case, merely the average value $\overline{F-1}$ over the time is required, it suffices to carry out the integration with regard to those members of (2) which do not depend on trigonometric functions. In this way we find

$$\begin{aligned}\overline{F-1} &= \left[\alpha'^2 + \frac{\alpha'^4}{2} + \frac{3}{8}\alpha'^6 + \frac{5}{16}\alpha'^8 - \frac{\alpha'^2}{2} - \frac{3}{8}\alpha'^4 \right.\\ &\quad \left. - \frac{5}{16}\alpha'^6 - \frac{35}{128}\alpha'^8\right]\mu^2 \cdot \frac{1}{l}\int_0^l r^2\,dr \\ (8) \qquad &= \frac{\mu^2\alpha'^2 l^2}{6}\left(1 + \frac{1}{4}\alpha'^2 + \frac{1}{8}\alpha'^4 + \frac{5}{64}\alpha'^6 + \ldots\right) \\ &= \frac{\alpha'^2 \cdot \beta^2 \cdot \pi^2}{6}\left(1 + \frac{1}{4}\alpha'^2 + \frac{1}{8}\alpha'^4 + \frac{5}{64}\alpha'^6 + \ldots\right).\end{aligned}$$

As an example we may consider a jet-wave for which $\alpha = tg\,\theta_m = 0.5$ or $\alpha' = \sin\theta_m = 0.447$. With $\beta = 2$ we find from (8) $\overline{F-1} = 1.74$ while for a rectangular wave the expression (10) paragraph 3 gives 1.72. There is thus only a slight difference between the two types of waves with regard to resistance.

A good many applications may be made of the dependency between the resistance of a jet-wave and the quantities on which the said resistance depends. The general character of the said applications may be thus elucidated. The resistance of the wave out to a given distance l is determined by

$$(1) \qquad R = \overline{F} \cdot l \cdot r,$$

r being the resistance of one cm of the original jet. On the other hand we have approximately

$$(2) \qquad \overline{F} = 1 + \frac{\alpha'^2\beta^2\pi^2}{6}$$

or roughly, if the second member is tolerably great compared to 1,

$$(3) \qquad \overline{F} = \frac{\alpha'^2\beta^2\pi^2}{6}.$$

Now β is, with a constant value of l, proportional to the frequency $n = \frac{1}{T}$ and so it is seen from (1) and (3) that

$$\frac{\Delta R}{R} = 2\frac{\Delta n}{n} \tag{4}$$

i. e. a certain percentage change in the frequency gives rise to double the percentage change in resistance. A change in frequency may thus be measured through the corresponding change in resistance. (Jet-wave frequencymeter). With constant frequency and constant field a change in the current producing the jet will produce double the percentage change in the resistance as α' is proportional to the said current. A more complex system is indicated in fig. 29. The wave is produced through the interaction between a constant alternating current, supplied by the source V_A, and a constant field produced by the magnet M. The latter has two windings of which the one may be fed from the storage cell or battery B, while the other may be inserted in a d. c. circuit I. Now if the current in the latter circuit is raised, this will, according to the direction of the current in the winding on the magnet, give rise either to an increase or a decrease of the resistance in the wave between the two electrodes E_1 and E_2. So by the current in I we are able to control the resistance of another circuit II. Obviously a good many combinations of the kind in fig. 29 are possible. It may be noted that the resistance between two electrodes such as E_1 and E_2 is given by

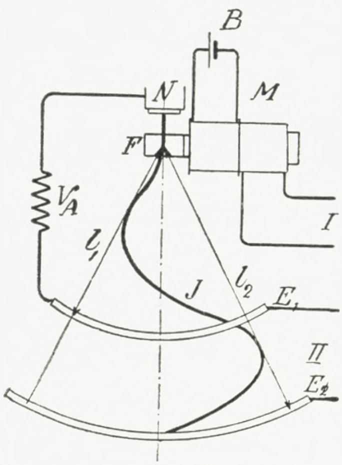


Fig. 29. Control of Resistance through magnetising Current.

$$R = rl_2 \cdot \overline{F}(\beta_2) - rl_1 \overline{F}(\beta_1) \tag{5}$$

thus approximately by

$$R = r(l_2 - l_1) + r \cdot \frac{\pi^2 \alpha'^2}{6} (\beta_2^2 - \beta_1^2). \tag{6}$$

6. Resistance between an Electrode in the Axis of the Wave and an Electrode perpendicular to the said Axis.

In fig. 30 a E and E_2 represent two adjacent electrodes, the one in the axis of the jet-wave, J, the other perpendicular to the said axis. The jet-wave will connect E and E_2 during the passage of every second half-wave. We shall

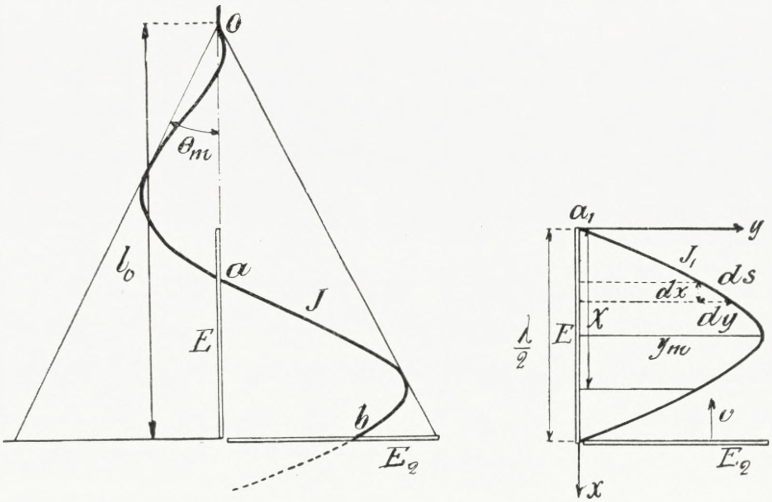


Fig. 30 a—b. Resistance of Wedge-Commutator.

endeavour to derive a formula for the average resistance taken over the time of passage of the part of the wave between E and E_2, the problem being of considerable interest in connection with certain practical applications of the wave. (Resistance of the Wedge Commutator).

In order to simplify the problem we shall replace the wave in fig. 30 a by a simple sine-shaped wave of constant amplitude, J_1, fig. 30 b. In the position shown the said wave may be represented by

$$(1) \qquad y = y_m \sin \mu x .$$

The resistance of the element ds may be written, compare paragraph 3,

$$dR = \frac{k}{S_0}\left[1 + \left(\frac{dy}{dx}\right)^2\right] dx, \tag{2}$$

S_0 indicating the area of the cross-section of the original jet. From (1) we get

$$dR = \frac{k}{S_0}(1 + y_m^2 \mu^2 \cos^2 \mu x)\, dx. \tag{3}$$

Thus the resistance from the origin of the wave a_1 to the abscissa x is

$$\begin{aligned} R &= \frac{k}{S_0}\int_0^x (1 + y_m^2 \mu^2 \cos^2 \mu x)\, dx \\ &= \frac{k}{S_0}\cdot x + \frac{k}{S_0}\frac{y_m^2 \mu^2}{2} x + \frac{k}{S_0}\frac{y_m^2 \mu^2}{4\mu}\sin(2\mu x). \end{aligned} \tag{4}$$

In order to obtain the average value in question we now put $x = vt$ and form the integral

$$\overline{R} = \frac{1}{\left(\frac{T}{2}\right)}\cdot\frac{k}{S_0}\int_0^{\frac{T}{2}}\left(vt + \frac{y_m^2 \mu^2}{2} vt + \frac{y_m^2 \mu^2}{4\mu}\sin 2\mu vt\right) dt. \tag{5}$$

This is the same as to assume that the electrode E_2 moves with the velocity of the jet upwards while the wave is kept in the position shown in fig. 30b. The last member in the brackets does not contribute to $\overline{R}$. Hence

$$\overline{R} = \frac{k}{S_0}\cdot\frac{\lambda}{2}\cdot\frac{1}{2}\left(1 + \frac{y_m^2 \mu^2}{2}\right). \tag{6}$$

Thus, if F now indicates the ratio of $\overline{R}$ and the resistance $\frac{k}{S_0}\cdot\frac{\lambda}{2}$ of the length $\frac{\lambda}{2}$ of the original jet, we have

$$F = \frac{1}{2}\left(1 + \frac{y_m^2 \mu^2}{2}\right). \tag{7}$$

Introducing the notations $\frac{y_m}{l} = \alpha$ and $\frac{l}{\left(\frac{\lambda}{2}\right)} = \beta$ and remembering that $\mu = \frac{2\pi}{\lambda}$, we may write

$$F = \frac{1}{2}\left(1 + \frac{\pi^2 \alpha^2 \beta^2}{2}\right). \tag{8}$$

In the application of (8) to the actual wave in fig. 30a the question is what value should be ascribed to β. It has been found that good agreement between observed and calculated values is established if β is determined as

$$\beta = \frac{l_0}{\left(\frac{\lambda}{2}\right)} - \frac{1}{4} = \beta_0 - \frac{1}{4}, \tag{9}$$

β_0 being the distance from the starting-point O of the wave to the electrode E_2 measured in half-wave-lengths. Thus for a wave $\alpha = 0.4$, $\beta = 1.8$, $d = 4.20$ mm (diameter of jet) the resistance here considered was found by measurement to be 5.0 milli-ohm, while from (8) and (9) was found 5.4 milli-ohm. The agreement was quite sufficient for the application. Actually the wave was not of the rectangular type indicated in fig. 30a but of the circular type. On the other hand, the electrode E_2 was approximately bent according to a circle with its centre in O. Judging from the comparison at the end of paragraph 5, it seems justifiable to assume that if (8) and (9) hold good for a rectangular wave in combination with a straight electrode E_2 as in fig. 30a, they may also be used in the case of the actual combination described.

7. Temperature-Gradient in a Jet carrying an electric Current.

In a jet carrying an electric current the temperature will rise from the point at which the current is introduced, say the nozzle, in the direction of the flow. An element Δx of the jet will commence heating at the moment it enters the current-carrying part of the jet, and it will go on heating as long as it is moving inside the said part. Its temperature must therefore increase steadily which means that the temperature of the jet must rise in the direction of the motion. The amount of heat dQ stored up in the element Δx during the time dt is

$$dQ = 0.239\, I^2 k \frac{\Delta x}{S} \cdot dt \quad \text{(g. cal.)}, \tag{1}$$

I being the current in Amp., k the specific resistance of the liquid in Ohm per cm/cm^2, and S the area of the cross-section of the jet in cm^2. The corresponding rise of temperature $d\vartheta$ is accordingly given by

$$c\varrho S \cdot \Delta x \cdot d\vartheta = 0.239\, I^2 k \cdot \frac{\Delta x}{S}\, dt, \tag{2}$$

c being the specific heat and ϱ the density of the liquid. During the interval dt the jet particle proceeds by the distance dx where

$$dx = v\, dt, \tag{3}$$

v being the velocity of the jet in cm/sec. Hence the rise of temperature along a piece dx of the jet is

$$d\vartheta = 0.239 \frac{k}{c\varrho} \cdot \frac{I^2}{vS^2} \cdot dx \tag{4}$$

from which follows that the temperature gradient (rise per cm) is

$$\dot{\vartheta} = 0.239 \cdot \frac{k}{c\varrho} \cdot \frac{I^2}{vS^2}. \tag{5}$$

With a mercury jet $k = 0.958 \cdot 10^{-4}$, $c = 0.033$, $\varrho = 13.6$ from which

(6) $$\dot{\vartheta} = 5.10 \cdot 10^{-5} \cdot \frac{I^2}{v S^2} \quad \text{(Centigrade/cm, Amp., cm/sec, cm}^2\text{)}.$$

The expression (5) is derived without the conductivity and the radiation of heat being taken into account. Estimates of the effect of the said factors show that in most cases they are practically insignificant. In a test of the theory the liquid from the jet was collected in a simple calorimeter and its temperature measured. With a mercury jet of 1.5 mm, a velocity of 253 cm/sec and a current of 20 Amp., the two values 0.213 and 0.232 centigrades were found for $\dot{\vartheta}$ while the value 0.257 centigrades is derived from (6). The difference between the observed and calculated values may easily be explained by the great difficulties of the measurement.

8. Heating of a Jet-Wave of rectangular Type.

In the most important application of the jet-wave, that of the jet-wave commutator, heavy currents are transmitted through the wave and it is thus the heating of the same which is of interest. We shall consider a particle Δs, fig. 31, which originates from a length Δx of the jet. The said particle is emitted from the centre O of the field at the moment t_0 in the direction θ. At the moment t it has reached the plane x as Δs. We shall assume the wave to have been produced by a current

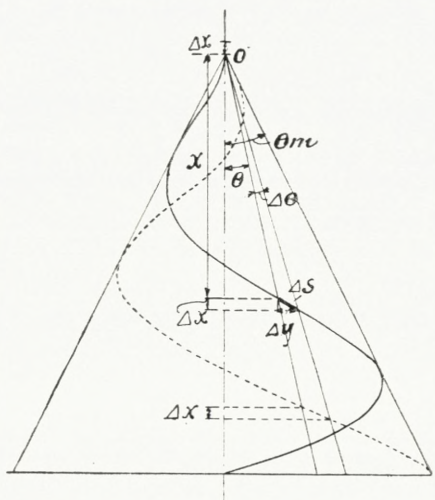


Fig. 31. Heating of a Jet-Wave, rectangular Type.

$$(1) \qquad i = I_0 \sin \omega t.$$

The direction θ of the path of the particle in question is, compare chapt. I, determined by

$$(2) \qquad tg\,\theta = \alpha \sin \omega t_0,$$

where $\alpha = tg\,\theta_m$. The length Δs of the particle is determined by

$$(3) \qquad \Delta s^2 = \Delta x^2 + \Delta y^2,$$

where the connection between Δx and Δy is given by the equation of the wave at the moment t. The latter is

$$(4) \qquad y = \alpha x \sin \omega \left(t - \frac{x}{v}\right).$$

Hence

$$(5) \qquad \Delta y = \left[\alpha \sin \omega \left(t - \frac{x}{v}\right) - \alpha \cdot \frac{x\,\omega}{v} \cos \omega \left(t - \frac{x}{v}\right)\right] \Delta x$$

or, as

$$(6) \qquad x = v\,(t - t_0)$$

$$(7) \qquad \Delta y = \left[\alpha \sin \omega t_0 - \alpha \frac{x\,\omega}{v} \cos \omega t_0\right] \Delta x.$$

Inserting in (3) we find

$$(8) \qquad \Delta s^2 = \left[1 + \left(\alpha \sin \omega t_0 - \alpha \frac{x\,\omega}{v} \cos \omega t_0\right)^2\right] \Delta x^2.$$

The resistance of Δs is

$$(9) \qquad \Delta R = \frac{k}{S} \Delta s,$$

where the area of the cross-section S of Δs is determined by

$$(10) \qquad S \cdot \Delta s = S_0 \Delta x,$$

S_0 being the area of cross-section of the original jet. Thus

$$(11) \qquad \Delta R = \frac{k}{S_0 \Delta x} \cdot \Delta s^2 = \frac{k}{S_0}\left[1 + \left(\alpha \sin \omega t_0 - \alpha \frac{x\,\omega}{v} \cos \omega t_0\right)^2\right] \Delta x.$$

The amount of heat dQ accumulated in Δs during the time $dt = \frac{dx}{v}$ is

$$
\begin{aligned}
& dQ = 0.239\, I^2 \Delta R \cdot dt \\
(12) \quad & \quad = 0.239 \frac{I^2 k}{S_0} \left[1 + \left(\alpha \sin \omega t_0 - \alpha \frac{x\,\omega}{v} \cos \omega t_0 \right)^2 \right] \Delta x \cdot \frac{dx}{v}
\end{aligned}
$$

and the corresponding rise of temperature $d\vartheta$ is determined by

(13) $$c \varrho S_0 \Delta x \cdot d\vartheta = dQ$$

from which

(14) $$d\vartheta = 0.239 \cdot \frac{I^2 k}{c \varrho S_0^2 v} \left[1 + \left(\alpha \sin \omega t_0 - \alpha \frac{x\,\omega}{v} \cos \omega t_0 \right)^2 \right] dx.$$

The rise of temperature obtained during the passage from the centre of the field to the distance x is found by integration of (14) from 0 to x. The result is

(15) $$\vartheta = \frac{0.239\, I^2 k}{c \varrho S_0^2 v} \left[x + x\,\alpha^2 \sin^2 \omega t_0 - x^2 \frac{\omega}{v} \alpha^2 \sin \omega t_0 \cos \omega t_0 + \frac{1}{3} x^3 \frac{\alpha^2 \omega^2}{v^2} \cos^2 \omega t_0 \right].$$

For the particle emitted at the moment $t_0 = 0$, thus the particle travelling along the axis of the wave, we get

(16) $$\vartheta = \frac{0.239\, I^2 k}{c \varrho S_0^2 v} \left[x + \frac{1}{3} \frac{\alpha^2 \pi^2}{\left(\frac{\lambda}{2}\right)^2} \cdot x^3 \right].$$

Here the first member represents the rise of temperature ϑ_0 in a non-undulating jet of length x. To this rise is added an amount $\Delta\vartheta$ which, measured in relation to ϑ_0, is

(17) $$\frac{\Delta\vartheta}{\vartheta_0} = \frac{1}{3} \pi^2 \alpha^2 \beta^2,$$

β standing for $\frac{x}{\left(\frac{\lambda}{2}\right)}$. If for instance $\alpha = 0.5$, $\beta = 2$ then

$\frac{\Delta \vartheta}{\vartheta_0} = \frac{\pi^2}{3} = 3.3$. The rise of temperature along the axis is thus more than four times that in the original jet carrying the same current.

For the particle travelling along the path of greatest deviation, thus corresponding to $\omega t_0 = \frac{\pi}{2}$, is found

$$(18) \qquad \vartheta = \frac{0.239\, I^2 k}{c \varrho S_0^2 v}(x + \alpha^2 x) = \frac{0.239\, I^2 k}{c \varrho S_0^2 v} \cdot \frac{x}{\cos^2 \theta_m}.$$

One would expect to find a rise of temperature equal to that in a straight jet of length $\frac{x}{\cos \theta_m}$, velocity $\frac{v}{\cos \theta_m}$ and area of cross-section $S_0 \cos \theta_m$. Actually the formula gives the value anticipated.

The rise of temperature in a piece of the wave between two planes perpendicular to the axis of the wave at the distances x_1 and x_2 from the origin of the said wave, is of course calculated as the difference between the values found for ϑ from the formula above by inserting x_2 and x_1 for x.

9. Heating of a Wave of circular Type.

Finally we shall consider the heating of a wave of the circular type, fig. 32. Again we shall imagine the wave to be produced by a current

$$(1) \qquad i = I_0 \sin \omega t$$

and we shall fix our attention on a particle Δs originating from a member Δr of the non-undulating jet. The said particle may pass the centre of the field at the moment t_0. It will then be sent out in a direction given by

$$(2) \qquad \sin \theta = \sin \theta_m \cdot \sin \omega t_0.$$

The length Δs of the wave-element formed by the particle at the moment t at the distance r from the origin of the wave is determined by

(3) $$\Delta s^2 = r^2 \Delta \theta^2 + \Delta r^2.$$

The connection between $\Delta \theta$ and Δr is found from the equation of the wave

(4) $$\sin \theta = \sin \theta_m \sin \omega \left(t - \frac{r}{v} \right)$$

which gives

(5) $$\Delta r = - \frac{\cos \theta}{\cos \omega t_0} \cdot \frac{v}{\omega} \cdot \frac{1}{\sin \theta_m} \cdot \Delta \theta = A \Delta \theta$$

it being noted that

(6) $$r = v(t - t_0).$$

From (3) and (5) is found

(7) $$\Delta s^2 = (r^2 + A^2) \Delta \theta^2.$$

The resistance in Δs is

(8) $$\Delta R = k \frac{\Delta s}{S},$$

where

(9) $$S \Delta s = S_0 \Delta r.$$

From (7), (8) and (9) is derived

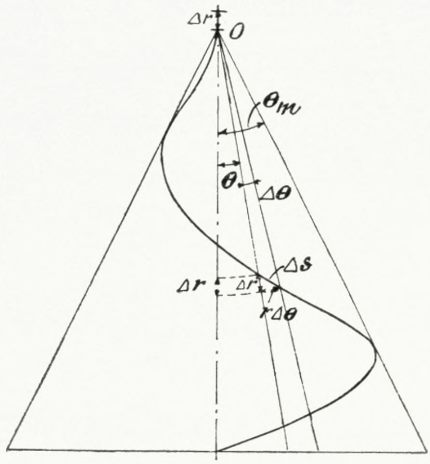


Fig. 32. Heating of a Jet-Wave, circular Type.

(10) $$\Delta R = \frac{k}{S_0 A} (r^2 + A^2) \Delta \theta.$$

During the motion through the distance dr, taking the time $dt = \frac{dr}{v}$, a quantity of heat dQ is stored up in the element Δs, where

(11) $$dQ = 0.239 \, I^2 \Delta R \cdot \frac{dr}{v}.$$

Introducing for ΔR (10) and identifying dQ with

(12) $$dQ = c\varrho S_0 \Delta r \cdot d\vartheta$$

we get

(13) $$d\vartheta = \frac{0.239\, I^2 k}{c\varrho S_0^2 v}\left(1 + \frac{r^2}{A^2}\right) dr.$$

The total rise of the temperature during the passage of the particle from the origin of the wave out to the distance r thus becomes

(14) $$\vartheta = \frac{0.239\, I^2 k}{c\varrho S_0^2 v}\, r \left(1 + \frac{1}{3}\frac{r^2}{A^2}\right).$$

Indicating by ϑ_0 the rise of temperature in a length l of the original jet we see from (14) that $\vartheta = \vartheta_0 + \Delta\vartheta$ where

(15) $$\frac{\Delta\vartheta}{\vartheta_0} = \frac{1}{3}\frac{r^2}{A^2} = \frac{1}{3}\left(\frac{\omega \cos \omega t_0 \cdot \sin \theta_m}{v \cos \theta}\right)^2 r^2,$$

where it should be noted that θ and t_0 are interconnected through (2). For $t_0 = 0$, $\theta = 0$, we get

(16) $$\frac{\Delta\vartheta}{\vartheta_0} = \frac{1}{3}\frac{\pi^2 r^2 \sin^2 \theta_m}{\left(\frac{\lambda}{2}\right)^2} = \frac{1}{3}\pi^2 \alpha'^2 \beta^2,$$

which should be compared with (17) in the preceding paragraph. With $t_0 = \frac{T}{4}$ the rise of temperature in the outermost particle of the wave is found. It is seen that $\Delta\vartheta = 0$, that is to say, the said particle is heated as much as a particle of the original jet would be. This result might be anticipated since the outermost particle does not suffer any deformation.

Again the heating of the wave between two concentric electrodes with radii r_2 and r_1 is found as the difference between the values ϑ_2 and ϑ_1 derived from (14) by introducing r_2 and r_1 for r.

APPENDIX

Experimental Test of the Theory of the Jet-Wave.

1. The Wave-Length.

According to the theory set forth above the length of a half-wave should, subject to certain conditions, be determined by

$$\frac{\lambda}{2} = v \cdot \frac{T}{2} = \frac{v}{2p}, \tag{1}$$

where T is the period and p the frequency of the alternating current used in the production of the wave, while v is the velocity of the jet. In order to test (1) a fairly large number of instantaneous pictures of the jet-wave was produced. Fig. 15 in chapt. II originates from this investigation. The wave-picture is seen against a plate of frosted glass on which the axis of the wave and the boundaries of the wave-space corresponding to the angular amplitude $\alpha = tg\,\theta_m = 0.5$ are drawn. The scale on the axis indicates the distance from the centre of the field in cm, thus the distance from the starting-point of the wave.

Table I.

h_0 = Distance from Surface of Mercury in Reservoir to the Jet-Hole.
d_0 = Diameter of Jet-Hole.
x_n = Distance from Jet-Hole to Number n Zero-Point of the Wave-Picture.

$$h = h_0 + \frac{x_{n+1} + x_n}{2}.$$

$$\left(\frac{\lambda}{2}\right)_c = v \cdot \frac{T}{2} + \Delta_p \frac{\lambda}{2} + \Delta_f \frac{\lambda}{2}.$$

$\left(\frac{\lambda}{2}\right)_0$ = Observed Half-Wave.

f_λ = Factor reducing the Scale to the Plane of the Wave.

Table I.

Plate No.	1. h_0 cm.	2. d_0 mm.	3. $\frac{\omega}{2\pi}$ sec.$^{-1}$	4. f_λ	5. x_n cm.	6. $\frac{x_{n+1}+x_n}{2}$ cm.	7. $x_{n+1}-x_n$ cm.	8. h cm.	9. $\frac{\pi\sqrt{2gh}}{\omega}$ cm.	10. $\Delta_p \frac{\lambda}{2}$ cm.	11. $\Delta_f \frac{\lambda}{2}$ cm.	12. $\left(\frac{\lambda}{2}\right)_c$ cm.	13. $\left(\frac{\lambda}{2}\right)_0$ cm.	14. $\frac{\left(\frac{\lambda}{2}\right)_0}{\left(\frac{\lambda}{2}\right)_c}$
55	228.0	6	50.0	0.991	5.57	9.13	7.11	237.1	6.82	— 0.04	+ 0.10	6.88	7.04	1.022
					12.68	16.15	6.94	244.2	6.93	— 0.04	0	6.89	6.87	0.997
					19.62	23.22	7.20	251.2	7.02	— 0.04	0	6.98	7.13	1.020
					26.82					(0.6 per cent.)				1.013
66	228.0	4.4	51.0	0.991	6.16	9.61	6.89	237.6	6.69	— 0.02	+ 0.15	6.82	6.82	1.000
					13.05	16.60	7.09	244.6	6.80	— 0.02	0	6.78	7.02	1.034
					20.14	23.65	7.02	251.7	6.89	— 0.02	0	6.87	6.95	1.011
					27.16					(0.3 per cent.)				1.015
69	228.0	5 [1]	51.0	0.991	4.90	8.30	6.79	236.3	6.68	— 0.03	+ 0.14	6.79	6.72	0.990
					11.69	15.12	6.86	243.1	6.78	— 0.03	0	6.75	6.79	1.005
					18.55	22.02	6.93	250.0	6.87	— 0.03	0	6.84	6.86	1.002
					25.48					(0.5 per cent.)				0.999
136	145.0	5	50.0	0.962	4.38	7.27	5.77	152.3	5.47	— 0.01	+ 0.07	5.53	5.55	1.002
					10.15	13.10	5.90	158.1	5.58	— 0.01	0	5.57	5.67	1.017
					16.05					(0.2 per cent.)				1.009

137	80.5	5	50.1	0.962	3.50	5.73	4.46	86.2	4.10	— 0	+ 0.05	4.15	4.29	1.033
					7.96	10.21	4.50	90.7	4.21	— 0	0	4.21	4.33	1.028
					12.46	14.76	4.59	95.3	4.31	— 0	0	4.31	4.41	1.023
					17.05	19.37	4.65	99.9	4.42	— 0	0	4.42	4.47	1.011
					21.70					(< 0.1 per cent.)				1.024
145 a	59.5	5	50.9	0.962	4.59	6.53	3.87	66.0	3.53	— 0	+ 0.04	3.57	3.72	1.042
					8.46	10.42	3.91	69.9	3.64	— 0	0	3.64	3.76	1.032
					12.37	14.42	4.10	73.9	3.74	— 0	0	3.74	3.94	1.053
					16.47					(< 0.1 per cent.)				1.042
145 b	59.5	5	50.9	0.962	3.39	5.39	4.00	64.9	3.50	— 0	+ 0.04	3.54	3.84	1.085
					7.39	9.38	3.98	68.9	3.61	— 0	0	3.61	3.82	1.058
					11.37	13.37	3.99	72.9	3.71	— 0	0	3.71	3.83	1.032
					15.36					(< 0.1 per cent.)				1.058
144 a	59.5	5	50.9	0.962	5.32	7.25	3.86	66.8	3.55	— 0	+ 0	3.55	3.71	1.045
					9.18	11.11	3.85	70.6	3.66	— 0	0	3.66	3.70	1.011
					13.03	15.07	4.07	74.6	3.76	— 0	0	3.76	3.91	1.039
					17.10	19.18	4.16	78.7	3.86	— 0	0	3.86	4.00	1.036
					21.26					(< 0.1 per cent.)				1.033
144 b	59.5	5	50.9	0.962	3.80	5.26	3.92	64.8	3.50	— 0	+ 0	3.50	3.77	1.076
					(6.72)	9.18	3.92	68.7	3.61	— 0	0	3.61	3.77	1.043
					11.64	13.61	3.93	73.1	3.72	— 0	0	3.72	3.78	1.016
					15.57	17.62	4.09	77.1	3.82	— 0	0	3.82	3.93	1.028
					19.66					(< 0.1 per cent.)				1.041

[1]) Cylindrical on the outmost 4 mm. of the bore.

In tab. I half-waves found from the photographs are compared with values of $\frac{\lambda}{2}$ derived from (1). The agreement is fairly good. The observed wave-length seems, however, to be slightly in excess of the wave calculated from (1). The fact is explained below by the influence of the surface-tension on the velocity of the wave. In the table certain corrections are indicated in columns 10 and 11. They will be understood from the following discussion of the test.

In the latter the half-wave read on the scale of the photographic plate is of course corrected for the small distance from the wave to the plate. It is the wave corrected in this way which is given in column 13. The velocity of the jet is determined by means of Torricelli's expression

$$(2) \qquad v = \sqrt{2gh}.$$

In (2) the influence of the surface-tension on the velocity of the jet proper is neglected, which, however, is justifiable[1]. Neither is the pressure-drop in the jet-pipe taken into account. It may be found by means of the Osborne Reynolds' law of similarity, according to which the pressure-drop $\dot{h}$ per cm of the pipe and measured in cm liquid-column is determined by

$$(3) \qquad \dot{h} = \frac{4v^2}{gd} \cdot f\left(\frac{vd}{\nu}\right),$$

where v is the velocity of the flow in the tube in cm/sec, d the diameter of the pipe in cm, ν the dynamic viscosity of the liquid which for mercury is 0.00115 at 20 centigrades, while finally f stands for the universal Osborne Reynolds'[2]

[1] On the Influence of the Surface-Tension on the Efflux of a Liquid in Jet-Form. Phys. Rev. Vol. XX, p. 728. 1922.

[2] Compare: A Comparison between the Flow of Water and Mercury in Pipes etc. Memoires de l'Académie des Sciences et Lettres de Danemark, Copenhague, 8^{me} Série, t. X, Nr. 5, 1926.

function of which a picture is given in fig. 33. By means of the latter figure and (3) the pressure-drop in the jet-pipe is calculated, and so again the percentage correction to be applied to the calculated wave-length. We shall illustrate the determination of the correction by means of Pl. 55. With the latter the velocity of the jet just below the jet-hole was $\sqrt{2\cdot 981\cdot 228} = 668$ cm/sec. The diameter of the jet was very nearly 0.5 cm, the coefficient of con-

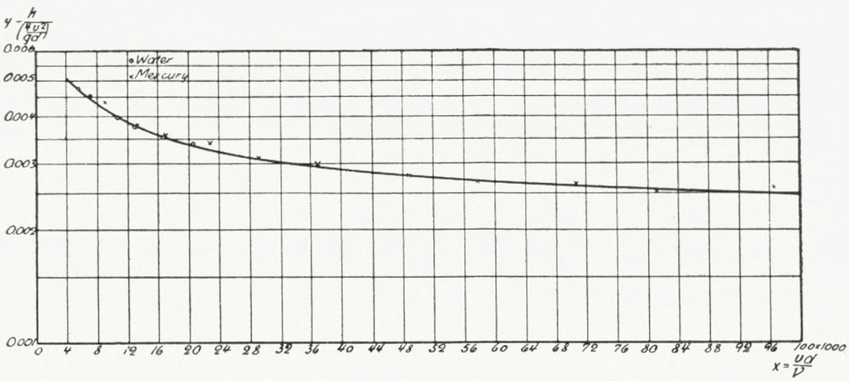


Fig. 33. Osborne Reynolds' Function $f\left(\frac{vd}{\nu}\right)$.

traction being 0.840. The internal diameter of the jet-pipe was 1.9 cm. The velocity v in the pipe is thus $\left(\frac{0.5}{1.9}\right)^2\cdot 668 = 46$ cm/sec. From this we find $\frac{vd}{\nu} = 76000$, and from the curve referred to above $f\left(\frac{vd}{\nu}\right) = 0.0026$, which again gives the value 0.0117 cm Hg for $\dot{h}$.

The length of the pipe was ab. 226 cm from which the total pressure-drop is found to be 2.65 cm Hg or $\frac{2.65}{226}\cdot 100 = 1.2$ per cent. of the head. It means that the velocity calculated from (2) must be reduced by 0.6 per cent. and the same is true for the wave-length. In tab. I the corrections found in this way are stated in column 10 under the heading $\Delta_p \frac{\lambda}{2}$.

A similar correction due to the pressure-drop in the jet-hole should furthermore be applied to the velocity and the wave-length. The correction is in all probability small, presumably below 1 per cent., judging from a special investigation. In the experiment Pl. 69 the bore was cylindrical of a length of ab. 4 mm. The effect hereof is traced in the small value of $\frac{\left(\frac{\lambda}{2}\right)_0}{\left(\frac{\lambda}{2}\right)_c}$. But in spite of the comparatively long bore the said quantity does not differ from the values corresponding to conical bores (55, 66) at the same head by more than ab. 1.5 per cent.

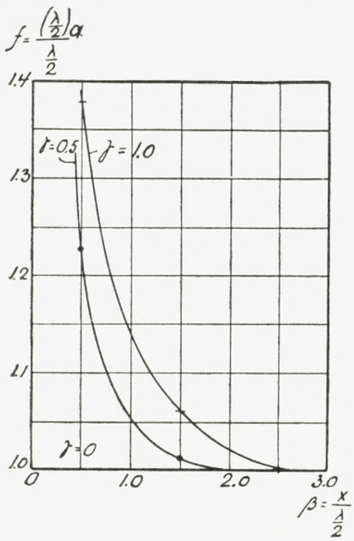


Fig. 34. Correction for the Distance to the Field.

Again it should be noted that the theory of the jet-wave shows that the half-waves close to the magnetic field should be somewhat longer than $v \cdot \frac{T}{2}$ and the longer, the greater the extension of the field in the direction of the jet. The correction to be applied to $v \cdot \frac{T}{2}$ may be taken from fig. 34. The abscissa indicates the distance from the centre of the field to the middle-point of the half-wave in question. The ordinate means the ratio of the half-wave predicted by the theory $\left(\frac{\lambda}{2}\right)a$ and the value $\frac{\lambda}{2}$ found as $v \cdot \frac{T}{2}$. The two curves correspond to two values of the length γ of the field. The length is measured with the half-wave $\frac{\lambda}{2}$ as unit, and strictly the length means the effective field-length which is a little different from the height of the pole-piece. In tab. I the correction found from fig. 34 is stated under $\Delta_f \frac{\lambda}{2}$ in column 11. The correction has obviously only bearing on the half-wave nearest to the field.

If hereupon the ratio of the observed and the calculated — and corrected — half-waves is formed (column 14), values close to 1 are found especially with higher heads. A systematic deviation, however, makes itself felt, the observed half-wave being, as already stated, greater than that calculated. There is some reason for believing that the discrepancy may be explained by the effect of the surface-tension on the velocity of the wave. It is known that a disturbance will travel along a cord of a mass per cm m and a tension P with a velocity

$$v = \sqrt{\frac{P}{m}}. \tag{4}$$

Now in the case of a cylindrical jet, produced from a liquid with the surface-tension C and the density ϱ, $P = C\pi d$ and $m = \varrho \cdot \frac{\pi}{4} d^2$, d being the diameter of the jet. Accordingly a deformation should run out along a mercury jet with a velocity

$$v_C = \sqrt{\frac{4C}{\varrho d}} = \frac{12.2}{\sqrt{d}}, \tag{5}$$

C being 500 c. g. s. and $\varrho = 13.55$. If now the velocity v_C in the case of the jet-wave is simply added to the velocity v of the jet, a half-wave should be anticipated which would be longer than the theoretical one by $\frac{v_C}{v} \cdot 100$ per cent.

In tab. II the values of $\left(\frac{v_C}{v}\right) \cdot 100$ are stated in column 4. They should be compared with the values in column 5 which show how great is the percentage excess of $\left(\frac{\lambda}{2}\right)_0$ over $\left(\frac{\lambda}{2}\right)_c$.

The two series of figures run parallelly, from which the conclusion may presumably be drawn that most of the discrepancy considered is actually due to the surface-tension. Obviously the correction for the latter would make the calculated half-wave about 1 per cent. greater than the observed. This remaining divergence may properly be explained by the pressure-drop in the jet-hole referred to above. So there is some reason for believing that a more exact test would prove the theory to hold good with a very high degree of accuracy with respect to its predictions as to the wave-length.

Table II.

Plate	1.	2.	3.	4.	5.
	d	v_C	v	$\left(\frac{v_C}{v}\right)\cdot 100$	$\left(\frac{\left(\frac{\lambda}{2}\right)_0}{\left(\frac{\lambda}{2}\right)_c} - 1\right)\cdot 100$
	cm.	cm./sec.	cm./sec.	p. c.	p. c.
55	0.504	17	690	2.5	1.3
66	0.370	20	680	2.9	1.5
69	0.420	19	680	2.8	— 0.1
136	—	—	550	3.5	0.9
137	—	—	420	4.5	2.4
145 a	—	—	390	4.9	4.2
145 b	—	—	360	5.3	5.8
144 a	—	—	370	5.1	3.3
144 b	—	—	370	5.1	4.1

2. The Amplitude of the Wave.

The approximate theory in Chapter II predicts for the amplitude of the jet-wave a value given by

$$\sin\theta_m = \frac{1}{10}\cdot\frac{IH}{m\,v^2}L_e\cdot\frac{\sin\gamma_e\frac{\pi}{2}}{\gamma_e\frac{\pi}{2}}. \tag{1}$$

It was shown that the said theory was in good agreement with the more exact theory for field-lengths up to $\frac{\lambda}{4}$ and for amplitudes up to $tg\,\theta_m = 0.5$. An experimental test on the validity of (1) is now recorded.

The test was made with 6 commutators in a three-phase series rectifier. The field-curves were known for the magnets so that the effective field-lengths could be calculated. They were in all cases 2.79 cm, while the height of the pole-pieces was 2.30 cm. The half-wave-length was 5.90 cm derived from the head $h = 177$ cm by means of the formula $v = \sqrt{2gh}$ and the frequency of the alternating current which was 50.0. The mass m per cm of the jets was calculated from the diameters d of the jets and from the density of mercury which was assumed to be 13.40, corresponding to a stationary temperature of 80 centigrades of the mercury under normal operation of the rectifier. The diameters d were again found from the diameters d_0 of the bores, the coefficients of contraction being determined for each of the bores by means of an experiment of efflux. The velocity of the jets was, as indicated, assumed to be that found from Torricelli's law. Obviously if the actual velocity is smaller than the velocity by one per cent., the experiment of efflux will give a value for the mass m per cm which is too small by one per cent., and the value for $m\,v^2$ used in the test will be too high by one per cent.

The result of the test is given in table III.

It is seen from the table that the observed values for θ_m are found to be on an average 2 per cent. greater than the values calculated from (1). The discrepancy could be explained by an error of 2 per cent. in the assumed value for v, due to friction in the jet-pipe and the nozzle. In all

Table III.

Com-mut. No.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
	I_A Amp.	H Gauss	d_0 cm.	$\varkappa_c$	$\sin\theta_m$ cal.	$\sin\theta_m$ cor.	$tg\,\theta_m$ obs.	θ_m obs.	$\sin\theta_m$ obs.	$f = \frac{9.}{6.}$
1	103	4880	0.445	0.844	0.349	0.358	0.399	21°43′	0.371	1.035
2	118	4450	0.446	0.840	0.367	0.376	0.418	22°42′	0.386	1.030
3	114	4650	0.444	0 855	0.360	0.370	0.395	21°33′	0.367	0.990
4	101	4995	0.450	0.826	0.357	0.366	0.396	21°36′	0.368	1.005
5	114.5	4810	0.451	0.836	0.380	0.389	0.443	23°54′	0.405	1.040
6	108	4790	0.450	0.842	0.352	0.361	0.390	21°21′	0.364	1.010
										1.018

probability there is an error which is substantially of this size. On the whole the expression (1) is seen to yield an excellent means for the calculation of the amplitude or of the current necessary for the production of a wave of given amplitude.

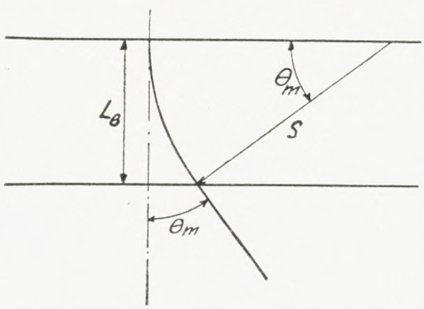


Fig. 35. Correction to Approximate Theory.

A small correction is indicated in column (6). The expression (1) is based on the assumption that it takes the time $\frac{L_e}{v}$ for any of the particles of the wave to pass the field. This, however, is not quite true. The outermost particle of the wave will, inside the field, follow a path which may approximately be considered a part of a circle as indicated in fig. 35. It will travel in this path with the velocity of the jet. Owing to this fact the value of L_e in (1) should be increased by $\frac{\theta_m^2}{6}\cdot 100$ per cent. or in the cases considered $\sin\theta_m$ cal. should be increased by the said percentage amount.

The correction may be derived as follows. From fig. 35 it appears that

(2) $$\varrho = \frac{L_e}{\sin \theta_m}$$

so that

(3) $$\varrho\theta_m = L_e \cdot \frac{\theta_m}{\sin \theta_m}$$

and

(4) $$\frac{\varrho\,\theta_m - L_e}{L_e} = \frac{\varDelta L_e}{L_e} = \frac{\theta_m - \sin \theta_m}{\sin \theta_m}.$$

Or with sufficient exactness from the series-development of $\sin \theta_m$

(5) $$\frac{\varDelta L_e}{L_e} = \frac{\theta_m^2}{6}.$$

CONTENTS

CHAPTER III

Particular Properties of the Jet-Wave.

APPENDIX

Experimental Test of the Theory of the Jet-Wave.

Færdig fra Trykkeriet den 31. Maj 1929.